S: Wegg. Div: Joh: Coll:
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EUCLID's ELEMENTS OF GEOMETRY,

FROM THE
Latin Tranſlation of COMMANDINE.

To which is added,
A TREATISE of the *Nature* of *Arithmetic* of
LOGARITHMS; Likewiſe
Another of the ELEMENTS of *Plain* and *Spherical*
TRIGONOMETRY; With
A PREFACE, ſhewing the *Uſefulneſs* and *Excellency* of
this WORK.

By Doctor *JOHN KEIL*, F. R. S. and late
Profeſſor of ASTRONOMY in *Oxford*.

The *Whole* Reviſed; where deficient, Supplied; where loſt
or corrupted, Reſtored. Alſo
Many Faults committed by Dr. HARRIS, Mr. CASWEL,
Mr. HEYNES, and other TRIGONOMETRICAL
WRITERS, are ſhewn; and in thoſe
Caſes where *They* are miſtaken, here are
given Solutions *Geometrically* True.
An Ample Account of which may be ſeen in the PREFACE;

By *SAMUEL CUNN*.

The FOURTH EDITION.

To which is ſubjoined an APPENDIX, containing the
Inveſtigation of thoſe Series's omitted by the AUTHOR.
And the Difference between Dr. KEIL and Mr.
CUNN impartially examined and adjuſted.

LONDON:
Printed for THO. WOODWARD, at the *Half-Moon*, between the *Two Temple-Gates* in *Fleet-Street*; and ſold by
J. OSBORN, at the *Golden Ball* in *Pater-noſter-Row.*
MDCCXLI.

Dr. *KEIL's*

PREFACE.

A *YOUNG Mathematician may be ſurpriſed, to ſee the old obſolete* Elements *of* Euclid *appear afreſh in Print; and that too after ſo many new Elements of Geometry, as have been lately publiſh'd; eſpecially ſince thoſe who gave us the Elements of Geometry, in a new manner, would have us believe they have detected a great many Faults in* Euclid. *Theſe acute* Philoſophers *pretend to have diſcovered, that* Euclid*'s Definitions are not perſpicuous enough; that his Demonſtrations are ſcarcely evident; that his whole Elements are ill-diſpos'd; and that they have found out innumerable Falſities in them, which had lain hid to their Times.*

But by their Leave, I make bold to affirm, that they carp at Euclid *undeſervedly: For his Definitions are diſtinct and clear, as being taken from firſt Principles, and our moſt eaſy and ſimple Conceptions; and his Demonſtrations elegant, perſpicuous and conciſe, carrying with them ſuch Evidence, and ſo much Strength of Reaſon, that I am eaſily induced to believe the Obſcurity* Scioliſts *ſo often accuſe* Euclid *with, is rather to be* *attributed to their own perplexed Ideas, than to the Demonſtrations themſelves. And however ſome may find Fault with the Diſpoſition and Order of his Elements, yet notwithſtanding I do not find any Method, in all the Writings of this kind, more proper and eaſy for Learners than that of* Euclid.

It is not my Buſineſs here to anſwer ſeparately every one of theſe Cavillers; *but it will eaſily appear to any one, moderately verſed in theſe Elements, that they rather ſhew their own Idleneſs, than any real Faults in* Euclid. *Nay, I dare venture to ſay, there is not one of theſe new Syſtems, wherein there are not more Faults, nay, groſſer Paralogiſms, than they have been able even to imagine in* Euclid.

After ſo many unſucceſsful Endeavours, in the Reformation of Geometry, ſome very good Geometricians, not daring to make new Elements, have deſervedly preferr'd Euclid *to all others; and have accordingly made it their Buſineſs to publiſh thoſe of* Euclid. *But they, for what Reaſon I know not, have entirely omitted ſome Propoſitions, and have altered the Demonſtrations of others for worſe. Among whom are chiefly* Tacquet *and* Dechalles, *both of which have unhappily rejected ſome elegant Propoſitions in the Elements, (which ought to have been retained) as imagining them trifling and uſeleſs;*

uſeleſs; ſuch, for Example, as Prop. 27, 28, *and* 29, *of the ſixth Book, and ſome others, whoſe Uſes they might not know. Farther, where-ever they uſe Demonſtrations of their own, inſtead of* Euclid's, *in thoſe Demonſtrations they are faulty in their Reaſoning, and deviate very much from the Conciſeneſs of the Antients.*

In the fifth Book, they have wholly rejected Euclid's *Demonſtrations, and have given a Definition of Proportion different from* Euclid's; *and which comprehends but one of the two Species of Proportion, taking in only commenſurable Quantities. Which great Fault no Logician or Geometrician would have ever pardoned, had not thoſe Authors done laudable Things in their other Mathematical Writings. Indeed, this Fault of theirs is common to all Modern Writers of Elements, who all ſplit on the ſame Rock; and to ſhew their Skill, blame* Euclid, *for what, on the contrary, he ought to be commended; I mean the Definition of proportional Quantities, wherein he ſhews an eaſy Property of thoſe Quantities, taking in both commenſurable and incommenſurable ones, and from which all the other Properties of Proportionals do eaſily follow.*

Some Geometricians, forſooth, want a Demonſtration of this Property in Euclid; *and undertake to ſupply the Deficiency by one of their own. Here, again,* *they ſhew their Skill in Logic, in requireing a Demonſtration for the Definition of a Term; that Definition of* Euclid *being ſuch as determines thoſe Quantities Proportionals which have the Conditions ſpecified in the ſaid Definition. And why might not the Author of the Elements give what Names he thought fit to Quantities having ſuch Requiſites? Surely he might uſe his own Liberty, and accordingly has called them Proportionals.*

But it may be proper here to examine the Method whereby they endeavour to demonſtrate that Property: Which is by firſt aſſuming a certain Affection, agreeing only to one kind of Proportionals, viz. *Commenſurables; and thence, by a long Circuit, and a perplexed Series of Concluſions, do deduce that univerſal Property of Proportionals which* Euclid *affirms; a Procedure foreign enough to the juſt Methods and Rules of Reaſoning. They would certainly have done much better, if they had firſt laid down that univerſal Property by* Euclid, *and thence have deduced that particular Property agreeing to only one Species of Proportionals. But rejecting this Method, they have taken the Liberty of adding their Demonſtration to this Definition of the fifth Book. Thoſe who have a Mind to ſee a farther Defence of* Euclid, *may*

may conſult the Mathematical Lectures of the learned Dr. Barrow.

As I have happened to mention this great Geometrician, I muſt not paſs by the Elements publiſhed by him, wherein generally he has retained the Conſtructions and Demonſtrations of Euclid *himſelf, not having omitted ſo much as one Propoſition. Hence, his Demonſtrations become more ſtrong and nervous, his Conſtructions more neat and elegant, and the Genius of the ancient Geometricians more conſpicuous, than is uſually found in other Books of this kind. To this he has added ſeveral Corollaries and Scholia, which ſerve not only to ſhorten the Demonſtrations of what follows, but are likewiſe of Uſe in other Matters.*

Notwithſtanding this, Barrow'*s Demonſtrations are ſo very ſhort, and are involved in ſo many Notes and Symbols, that they are rendered obſcure and difficult to one not verſed in Geometry. There many Propoſitions, which appear conſpicuous in reading* Euclid *himſelf, are made knotty and ſcarcely intelligible to Learners by this Algebraical Way of Demonſtration, As is, for Example,* Prop. 13. Book I. *and the Demonſtrations which he lays down in* Book II. *are ſtill more difficult:* Euclid *himſelf has done much better, in ſhewing their Evidence by the Contemplations of Figures, as in Geometry ſhould always be done. The Elements of all Sci-*

ences ought to be handled after the moſt ſimple Method, and not to be involv'd in Symbols, Notes, or obſcure Principles, taken elſewhere.

As Barrow's *Elements are too ſhort, ſo are thoſe of* Clavius *too prolix, abounding in ſuperfluous Scholiums and Comments: For in my Opinion* Euclid *is not ſo obſcure as to want ſuch a Lumber of Notes, neither do I doubt but a Learner will find* Euclid *himſelf eaſier than any of his Commentators. As too much Brevity in Geometrical Demonſtrations begets Obſcurity, ſo too much Prolixity produces Tedioušneſs and Confuſion.*

On theſe Accounts, principally, it was, that I undertook to publiſh the firſt ſix Books of Euclid, *with the* 11th *and* 12th, *according to* Commandinus's *Edition; the reſt I forbore, becauſe thoſe firſt-mentioned are ſufficient for underſtanding of moſt Parts of the Mathematics now ſtudied.*

Farther, for the Uſe of thoſe who are deſirous to apply the Elements of Geometry to Uſes in Life, we have added a Compendium of Plain *and* Spherical Trigonometry, *by means whereof Geometrical Magnitudes are meaſur'd, and their Dimenſion expreſſed in Numbers.*

J. KEIL.

Mr. *CUNN*'s

PREFACE,

Shewing the USEFULNESS *and* EXCELLENCY *of this* WORK.

DR. *KEIL*, in his Preface, hath ſufficiently declared how much eaſier, plainer, and more elegant, the Elements of Geometry written by *Euclid* are, than thoſe written by others; and that the Elements themſelves are fitter for a Learner, than thoſe publiſhed by ſuch as have pretended to comment on, ſymbolize, or tranſpoſe any of his Demonſtrations of ſuch Propoſitions as they intended to treat of. Then how muſt a Geometrician be amazed, when he meets with a Tract * of the 1ſt, 2d, 3d, 4th, 5th, 6th, 11th and 12th *Books* of the Elements, in which are omitted the Demonſtrations of all the Propoſitions of that moſt noble univerſal Matheſis, the 5th; on which the 6th, 11th, and 12th ſo much depend, that the Demonſtration of not ſo much as one Propoſition in them can be obtained without thoſe in the fifth!

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* *Vide* the laſt Edition of the *Engliſh* Tacquet.

The 7th, 8th, and 9th *Books* treat of ſuch Properties of Numbers which are neceſſary for the Demonſtrations of the 10th, which treats of Incommenſurables; and the 13th, 14th, and 15th, of the five *Platonic* Bodies. But though the Doctrine of Incommenſurables, becauſe expounded in one and the ſame Plane, as the firſt ſix Elements were, claimed by a Right of Order, to be handled before Planes interſected by Planes, or the more compounded Doctrine of Solids; and the Properties of Numbers were neceſſary to the Reaſoning about Incommenſurables: Yet becauſe only one Propoſition of theſe four Books, *viz.* the 1ſt of the 10th, is quoted in the 11th and 12th Books; and that only once, *viz.* in the Demonſtration of the 2d of the 12th; and that 1ſt Propoſition of the 10th, is ſupplied by a *Lemma* in the 12th: And becauſe the 7th, 8th, 9th, 10th, 13th, 14th, 15th Books have not been thought (by our greateſt Maſters) neceſſary to be read by ſuch as deſign to make natural Philoſophy their Study, or by ſuch as would apply Geometry to practical Affairs, Dr. *Keil*, in his Edition, gave us only theſe eight Books, *viz.* the firſt ſix, and the 11th and 12th.

And as he found there was wanting a Treatiſe of theſe Parts of the Elements, as they were written by *Euclid* himſelf; he publiſhed his Edition without omitting any of *Euclid*'s Demonſtrations, except two; one of which was a ſecond Demonſtration of the 9th Propoſition of the third Book; and the other a Demonſtration of that Property of Proportionals called *Converſion* (contained in a *Corollary* to the

19th Propoſition of the fifth Book); where inſtead of *Euclid*'s Demonſtration, which is univerſal, moſt Authors have given us only particular ones of their own. The firſt of theſe, which was omitted is here ſupplied: And that which was corrupted is here reſtored *.

And ſince ſeveral Perſons, to whom the Elements of Geometry are of vaſt Uſe, either are not ſo ſufficiently skilled in, or perhaps have not Leiſure, or are not willing to take the Trouble, to read the *Latin*; and ſince this Treatiſe was not before in *Engliſh*, nor any other which may properly be ſaid to contain the Demonſtrations laid down by *Euclid* himſelf; I do not doubt but the Publication of this Edition will be acceptable, as well as ſerviceable.

Such Errors, either typographical, or in the Schemes, which were taken Notice of in the *Latin* Edition, are corrected in this.

As to the Trigonometrical Tract annexed to theſe Elements, I find our Author, as well as Dr. *Harris*, Mr. *Caſwell*, Mr. *Heynes*, and others of the Trigonometrical Writers, is miſtaken in ſome of the Solutions.

That the common Solution of the 12th *Caſe* of Oblique Spherics is falſe, I have demonſtrated, and given a true one. *See Page* 319.

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* *Vide Page* 55, 197, of *Euclid*'s Works, publiſhed by Dr. *Gregory*.

In the Solution of our 9th and 10th *Caſes*, by other Authors called the 1ſt and 2d, where are given and ſought oppoſite Parts, not only the aforementioned Authors, but all others that I have met with, have told us that the Solutions are ambiguous; which Doctrine is, indeed, ſometimes true, but ſometimes falſe: For ſometimes the *Quæſitum* is doubtful, and ſometimes not; and when it is not doubtful, it is ſometimes greater than 90 Degrees, and ſometimes leſs: And ſure I ſhall commit no Crime, if I affirm, that no Solution can be given without a juſt Diſtinction of theſe Varieties. For the Solution of theſe *Caſes* ſee my Directions at *Pages* 321, 322.

In the Solution of our 3d and 7th *Caſes*, in other Authors reckoned the 3d and 4th, where there are given two Sides and an Angle oppoſite to one of them, to find the 3d Side, or the Angle oppoſite to it; all the Writers of Trigonometry that I have met with, who have undertaken the Solutions of theſe two, as well as the two following *Caſes*, by letting fall a Perpendicular, which is undoubtedly the ſhorteſt and beſt Method for finding either of theſe *Quæſita*, have told us, that the $\left\{\begin{array}{l}\text{Sum}\\ \text{Diference}\end{array}\right\}$ of the Vertical Angles, or Baſes, ſhall be the ſought Angle or Side, according as the Perpendicular falls $\left\{\begin{array}{l}\text{within;}\\ \text{without;}\end{array}\right\}$ which cannot be known, unleſs the Species of that unknown Angle, which is oppoſite to a given Side, be firſt known.

Here they leave us firſt to calculate that unknown Angle, before we ſhall know whether we are to take the Sum or the Difference of the Vertical Angles or Baſes, for the ſought Angle or Baſe: And in the Calculation of that Angle have left us in the dark as to its Species; as appears by my Obſervations on the two preceding *Caſes.*

The Truth is, the *Quæſitum* here, as well as in the two former *Caſes*, is ſometimes doubtful, and ſometimes not; when doubtful, ſometimes each Anſwer is leſs than 90 Degrees, ſometimes each is greater; but ſometimes one leſs, and the other greater, as in the two laſt-mentioned *Caſes.* When it is not doubtful, the *Quæſitum* is ſometimes greater than 90 Degrees, and ſometimes leſs. All which Diſtinctions may be made without another Operation, or the Knowledge of the Species of that unknown Angle, oppoſite to a given Side; or, which is the ſame thing, the falling of the Perpendicular within or without. For which ſee my Directions at *Pages* 324, 325.

In the Solution of our 1ſt and 5th *Caſes* called in other Authors, the 5th and 6th; where there are given two Angles, and a Side oppoſite to one of them, to find the 3d Angle, or the Side oppoſite to it; they have told us, that the $\left\{\begin{matrix}\text{Sum}\\ \text{Difference}\end{matrix}\right\}$ of the Vertical Angles, or Baſes, according as the Perpendicular falls $\left\{\begin{matrix}\text{within}\\ \text{without}\end{matrix}\right\}$ ſhall be the ſought Angle or Side; and that it is known whether the Perpendicular falls within or without, by the Affection of the given Angles.

Here they ſeem to have ſpoken as tho' the *Quæſitum* was always determined, and never ambiguous; for they have here determined whether the Perpendicular falls within or without, and thereby whether they are to take the Sum or the Difference of the Vertical Angles or Baſes for the ſought Angle or Side.

But notwithſtanding theſe imaginary Determinations, I affirm, that the *Quæſitum* here, as in the two *Caſes* laſt-mentioned, is ſometimes ambiguous, and ſometimes not; and that too, whether the Perpendicular falls within, or whether it falls without. See my Solutions of theſe two *Caſes* in *Page* 323.

The Determination of the 3d *Caſe* of Oblique Plain Triangles, ſee in *Page* 325.

SAM. CUNN.

EUCLID's ELEMENTS.

BOOK I.

DEFINITIONS.

I. *A POINT is that which hath no Parts or Magnitude.*

II. *A Line is Length, without Breadth.*

III. *The Ends (or Bounds) of a Line are Points.*

IV. *A Right Line is that which lieth evenly between its Points.*

V. *A Superficies is that which hath only Length and Breadth.*

VI. *The Bounds of a Superficies are Lines.*

VII. *A plain Superficies is that which lieth evenly between its Lines.*

VIII. *A plain Angle is the Inclination of two Lines to one another in the ſame Plane, which touch each other, but do not both lie in the ſame Right Line.*

IX. *If the Lines containing the Angle be Right ones, then the Angle is called a Right-lined Angle.*

X. *When a Right Line, ſtanding on another Right Line, makes Angles on either Side thereof,* *of, equal between themſelves, each of theſe equal Angles is a Right one, and that Right Line which ſtands upon the other, is called a Perpendicular to that whereon it ſtands.*

XI. *An Obtuſe Angle is that which is greater than a Right one.*

XII. *An Acute Angle is that which is leſs than a Right one.*

XIII. *A Term (or Bound) is that which is the Extreme of any thing.*

XIV. *A Figure is that which is contained under one or more Terms.*

XV. *A Circle is a plain Figure, contained under one Line, called the Circumference; to which all Right Lines, drawn from a certain Point within the Figure, are equal.*

XVI. *And that Point is called the Centre of the Circle.*

XVII. *A Diameter of a Circle is a Right Line drawn through the Centre, and terminated on both Sides by the Circumference, and divides the Circle into two equal Parts.*

XVIII. *A Semicircle is a Figure contained under a Diameter, and that Part of the Circumference of a Circle, cut off by that Diameter.*

XIX. *A Segment of a Circle is a Figure contained under a Right Line, and Part of the Circumference of the Circle [which is cut off by that Right Line.]*

XX. *Right-lined Figures are ſuch as are contained under Right Lines.*

XXI. *Three-ſided Figures are ſuch as are contained under three Lines.*

XXII. *Four-ſided Figures are ſuch as are contained under four.*

XXII. *Many-ſided Figures are thoſe that are contained under more than four Right Lines.*

XXIV. *Of three-ſided Figures, that is an Equilateral Triangle, which hath three equal Sides.*

XXV. *That an Iſoſceles, or Equicrural one, which hath only two Sides equal.*

XXVI. *And a Scalene one, is that which hath three unequal Sides.*

XXVII. *Alſo of three-ſided Figures, that is a Right-angled Triangle, which hath a Right Angle.*

XXVIII. *That an Obtuſe-angled one, which hath an Obtuſe Angle.*

XXIX. *And that an Acute-angled one, which hath three Acute Angles.*

XXX. *Of four-ſided Figures, that is a Square, whoſe four Sides are equal, and its Angles all Right ones.*

XXXI. *That an Oblong, or Rectangle, which is longer than broad; but its oppoſite Sides are equal, and all its Angles Right ones.*

XXXII. *That a Rhombus, which hath four equal Sides, but not Right Angles.*

XXXIII. *That a Rhomboides, whoſe oppoſite Sides and Angles only are equal.*

XXXIV. *All Quadrilateral Figures, beſides theſe, are called Trapezia.*

XXXV. *Parallels are ſuch Right Lines in the ſame Plane, which if infinitely produc'd both ways, would never meet.*

POSTULATES.

I. G*RANT that a Right Line may be drawn from any one Point to another.*

II. *That a finite Right Line may be continued directly forwards.*

III. *And that a Circle may be deſcribed about any Centre, with any Diſtance.*

AXIOMS.

I. *THINGS equal to one and the ſame Thing, are equal to one another.*

II. *If to equal Things are added equal Things, the Wholes will be equal.*

III. *If from equal Things, equal Things be taken away, the Remainders will be equal.*

IV. *If equal Things be added to unequal Things, the Wholes will be unequal.*

V. *If equal Things be taken from unequal Things, the Remainders will be unequal.*

VI. *Things which are double to one and the ſame Thing, are equal between themſelves.*

VII. *Things which are half one and the ſame Thing, are equal between themſelves.*

VIII. *Things which mutually agree together, are equal to one another.*

IX. *The Whole is greater than its Parts.*

X. *Two Right Lines do not contain a Space.*

XI. *All Right Angles are equal between themſelves.*

XII. *If a Right Line, falling upon two other Right Lines, makes the inward Angles on the ſame Side thereof, both together, leſs than two Right Angles, thoſe two Right Lines, infinitely produc'd, will meet each other on that Side where the Angles are leſs than Right ones.*

Note, When there are ſeveral Angles at one Point, any one of them is expreſs'd by three Letters, of which that at the Vertex of the Angle is plac'd in the Middle. For Example; In the Figure of *Prop.* XIII. *Lib.* I. the Angle contain'd under the Right Lines A B, B C, is called the Angle A B C; and the Angle contain'd under the Right Lines A B, B E, is called the Angle A B E.

PRO-

PROPOSITION I.

PROBLEM.

To deſcribe an Equilateral Triangle upon a given finite Right Line.

LET AB be the given finite Right Line, upon which it is required to deſcribe an Equilateral Triangle.

About the Centre A, with the Diſtance AB, deſcribe the Circle BCD*; and about the * 3 *Poſt.* Centre B, with the ſame Diſtance BA, deſcribe the Circle ACE; and from the Point C, where the two Circles cut each other, draw the Right Lines CA, CB†. † 1 *Poſt.*

Then becauſe A is the Centre of the Circle DBC, AC ſhall be equal to AB‡. And becauſe B is the ‡ 15 *Def.* Centre of the Circle CAE, BC ſhall be equal to BA: but CA hath been proved to be equal to AB; therefore both CA and CB are each equal to AB. But Things equal to one and the ſame Thing, are equal between themſelves, and conſequently CA is equal to CB; therefore the three Sides CA, AB, BC, are equal between themſelves.

And ſo the Triangle BAC is an Equilateral one, and is deſcribed upon the given finite Right Line AB; which was to be done.

PROPOSITION II.

PROBLEM.

At a given Point, to put a Right Line equal to a Right Line given.

LET the Point given be A, and the given Right Line BC; it is required to put a Right Line at the Point A, equal to the given Right Line BC.

* *Poſt.* 1. Draw the Right Line A C from the Point A to C*,
† 1 *of this.* upon it deſcribe the Equilateral Triangle D A C†;
produce D A and D C directly forwards to E and
‡ *Poſt.* 2. G‡; about the Centre C, with the Diſtance B C,
* *Poſt.* 3. deſcribe the Circle B G H*; and about the Centre D,
with the Diſtance D G, deſcribe the Circle G K L.

Now becauſe the Point C is the Centre of the Circle
† *Def.* 15. B G H, B C will be equal to C G†; and becauſe
D is the Centre of the Circle G K L, the Whole D L
will be equal to the Whole D G, the Parts whereof
D A and D C are equal; therefore the Remainders
‡ *Axiom* 3. A L, G C, are alſo equal‡. But it has been demonſtrated, that B C is equal to C G; wherefore both A L and B C are each of them equal to C G. But Things that are equal to one and the ſame Thing, are equal to one another; and therefore likewiſe A L is equal to B C.

Whence the Right Line A L is put at the given Point A, equal to the given Right Line B C, which was to be done.

PROPOSITION III.

PROBLEM.

Two unequal Right Lines being given, to cut off a Part from the greater Equal to the leſſer.

LET A B and C be the two unequal Right Lines given, the greater whereof is A B; it is required to cut off a Line from the greater A B equal to the leſſer C.

* 2 *of this.* Put* a Right Line A D at the Point A, equal to the
Line C; and about the Centre A, with the Diſtance
† *Poſt.* 3. A D, deſcribe a Circle D E F†.

Then, becauſe A is the Centre of the Circle D E F,
A E is equal to A D; and ſo both A E and C are
each equal to A D; wherefore A E is likewiſe equal
‡ *Axiom* 1. to C‡.

And ſo there is cut off from A B, the greater of two given Right Lines A B and C, a Line A E equal to the leſſer Line C; which was to be done.

PRO-

PROPOSITION IV.

THEOREM.

If there are two Triangles that have two Sides of the one equal to two Sides of the other, each to each, and the Angle contained by thoſe equal Sides in one Triangle equal to the Angle contained by the correſpondent Sides in the other Triangle, then the Baſe of one of the Triangles ſhall be equal to the Baſe of the other, the whole Triangle equal to the whole Triangle, and the remaining Angles of one equal to the remaining Angles of the other, each to each, which ſubtend the equal Sides.

LET the two Triangles be ABC, DEF, which have two Sides AB, AC, equal to two Sides DE, DF, each to each, that is, the Side AB equal to the Side DE, and the Side AC to DF; and the Angle BAC equal to the Angle EDF. I ſay, that the Baſe BC is equal to the Baſe EF, the Triangle ABC equal to the Triangle DEF, and the remaining Angles of the one equal to the remaining Angles of the other, each to its Correſpondent, ſubtending the equal Sides, *viz.* the Angle ABC equal to the Angle DEF, and the Angle ACB equal to the Angle DFE.

For the Triangle ABC being applied to DEF, ſo as the Point A may co-incide with D, and the Right Line AB with DE, then the Point B will co-incide with the Point E, becauſe AB is equal to DE. And ſince AB co-incides with DE, the Right Line AC likewiſe will co-incide with the Right Line DF, becauſe the Angle BAC is equal to the Angle EDF. Wherefore alſo C will co-incide with F, becauſe the Right Line AC is equal to the Right Line DF. But the Point B co-incides with E, and therefore the Baſe BC co-incides with the Baſe EF. For if the Point B co-inciding with E, and C with F, the Baſe BC does not co-incide with the Baſe EF; then two Right Lines will contain a Space, which is impoſſible*. ** Ax.* 10. Therefore the Baſe BC co-incides with the Baſe EF, and is equal thereto; and conſequently the whole Triangle ABC will co-incide with the whole Triangle DEF, and will be equal thereto; and the remaining Angles will co-incide with the remaining Angles†, and will be equal to them, *viz.* the Angle ABC equal to the Angle DEF, and the Angle ACB equal to the Angle DFE. Which was to be demonſtrated.

† *Ax.* 8.

PROPOSITION V.

THEOREM.

The Angles at the Baſe of an Iſoſceles *Triangle are equal between themſelves: And if the equal Sides be produced, the Angles under the Baſe ſhall be equal between themſelves.*

LET ABC be an *Iſoſceles* Triangle, having the Side AB equal to the Side AC; and let the equal Sides AB, AC, be produced directly forwards to D and E. I ſay, the Angle ABC is equal to the Angle ACB, and the Angle CBD equal to the Angle BCE.

For aſſume any Point F in the Line BD, and from AE cut off the Line AG equal * to AF, and join FC, GB.

* 3 *of this.*

Then, becauſe AF is equal to AG, and AB to AC, the two Right Lines FA, AC, are equal to the two Lines GA, AB, each to each, and contain the common Angle FAG; therefore the Baſe FC is equal † to the Baſe GB, and the Triangle AFC equal to the Triangle AGB, and the remaining Angles of the one equal to the remaining Angles of the other, each to each, ſubtending the equal Sides, *viz.* the Angle ACF equal to the Angle ABG; and the Angle AFC equal to the Angle AGB. And becauſe the Whole AF is equal to the Whole AG, and the Part AB equal to the Part AC, the Remainder BF is equal to the Remainder CG. But FC‡ has been proved to be equal to GB; therefore the two Sides BF, FC, are equal to the two Sides CG, GB, each to each, and the Angle BFC equal to the Angle CGB; but they have a common Baſe BC. Therefore alſo the Triangle BFC will be equal to the Triangle CGB, and the remaining Angles of the one equal

† 4 *of this.*

‡ *Ax.* 3.

equal to the remaining Angles of the other, each to each, which ſubtend the equal Sides. And ſo the Angle FBC is equal to the Angle GCB; and the Angle BCF equal to the Angle CBG. Therefore, becauſe the whole Angle ABG has been proved equal to the whole Angle ACF, and the Part CBG equal to BCF, the remaining Angle ABC will be * equal to the remaining Angle ACB; but theſe are the Angles at the Baſe of the Triangle ABC. It hath likewiſe been proved, that the Angles FBC, GCB, under the Baſe, are equal; therefore the Angles at the Baſe of Iſoſceles Triangles are equal between themſelves; and if the equal Right Lines be produced, the Angles under the Baſe will be alſo equal between themſelves.

* *Ax.* 3.

Coroll. Hence every Equilateral Triangle is alſo Equiangular.

PROPOSITION VI.

THEOREM.

If two Angles of a Triangle be equal, then the Sides ſubtending the equal Angles will be equal between themſelves.

LET ABC be a Triangle, having the Angle ABC equal to the Angle ACB. I ſay, the Side AB is likewiſe equal to the Side AC.

For if AB be not equal to AC, let one of them, as AB, be the greater, from which cut off BD equal to AC †, and join DC. Then, becauſe BD is equal to AC, and BC is common, DB, BC, will be equal to AC, CB, each to each, and the Angle DBC equal to the Angle ACB, from the Hypotheſis; therefore the Baſe DC is equal ‡ to the Baſe AB, and the Triangle DBC equal to the Triangle ACB, a Part to the Whole, which is abſurd; therefore AB is not unequal to AC, and conſequently is equal to it.

† 3 *of this.*

‡ 4 *of this.*

Therefore, if two Angles of a Triangle be equal between themſelves, the Sides ſubtending the equal Angles are likewiſe equal between themſelves. Which was to be demonſtrated.

Coroll. Hence every Equiangular Triangle is alſo Equilateral.

PROPOSITION VII.

THEOREM.

On the ſame Right Line cannot be conſtituted two Right Lines equal to two other Right Lines, each to each, at different Points, on the ſame Side, and having the ſame Ends which the firſt Right Lines have.

FOR, if it be poſſible, let two Right Lines AD, DB, equal to two others AC, CB, each to each, be conſtituted at different Points C and D, towards the ſame Parts CD, and having the ſame Ends A and B which the firſt Right Lines have, ſo that CA be equal to AD, having the ſame End A which CA hath; and CB equal to DB, having the ſame End B; and let CD be joined.

Then, becauſe AC is equal to AD, the Angle ACD will be equal * to the Angle ADC, and conſequently the Angle ADC is greater than the Angle BCD; wherefore the Angle BDC will be much greater than the Angle BCD. Again, becauſe CB is equal to DB, the Angle BDC will be equal to the Angle BCD; but it has been proved to be much greater, which is impoſſible. Therefore *on the ſame Right Line cannot be conſtituted two Right Lines equal to two other Right Lines, each to each, at different Points, on the ſame Side, and having the ſame Ends which the firſt Right Lines have*; which was to be demonſtrated.

* 5 *of this.*

PRO-

PROPOSITION VIII.

THEOREM.

If two Triangles have two Sides of the one equal to two Sides of the other, each to each, and the Baſes equal, then the Angles contained under the equal Sides will be equal.

LET the two Triangles be ABC, DEF, having two Sides AB, AC, equal to two Sides DE, DF, each to each, *viz.* AB equal to DE, and AC to DF; and let the Baſe BC be equal to the Baſe EF. I ſay, the Angle BAC is equal to the Angle EDF.

For if the Triangle ABC be applied to the Triangle DEF, ſo that the Point B may co-incide with E, and the Right Line BC with EF, then the Point C will co-incide with F, becauſe BC is equal to EF. And ſo, ſince BC co-incides with EF, BA and AC will likewiſe co-incide with ED and DF. For if the Baſe BC ſhould co-incide with EF, and at the ſame time the Sides BA, AC, ſhould not co-incide with the Sides ED, DF, but change their Poſition, as EG, GF, then there would be conſtituted on the ſame Right Line two Right Lines, equal to two other Right Lines, each to each, at ſeveral Points, on the ſame Side, having the ſame Ends. But this is proved to be otherwiſe*; therefore it is impoſſible for the Sides BA, AC, not to co-incide with the Sides ED, DF, if the Baſe BC co-incides with the Baſe EF; wherefore they will co-incide, and conſequently the Angle BAC will co-incide with the Angle EDF, and will be equal to it. Therefore, *if two Triangles have two Sides of the one equal to two Sides of the other, each to each, and the Baſes equal, then the Angles contained under the equal Sides will be equal*; which was to be demonſtrated.

* 7 *of this.*

PRO-

PROPOSITION IX.

PROBLEM.

To cut a given Right-lin'd Angle into two equal Parts.

LET BAC be a given Right-lin'd Angle, which is required to be cut into two equal Parts.

Aſſume any Point D in the Right Line AB, and
* 3 *of this.* cut off AE from the Line AC equal to AD*; join
† 1 *of this.* DE, and thereon make † the Equilateral Triangle DEF, and join AF. I ſay, the Angle BAC is cut into two equal Parts by the Line AF.

For, becauſe AD is equal to AE, and AF is common, the two Sides DA, AF, are each equal to the two Sides AE, AF, and the Baſe DF is equal to
‡ 8 *of this.* the Baſe EF; therefore ‡ the Angle DAF is equal to the Angle EAF. Wherefore *a given Right-lin'd Angle is cut into two equal Parts*; which was to be done.

PROPOSITION X.

PROBLEM.

To cut a given finite Right Line into two equal Parts.

LET AB be a given finite Right Line, required to be cut into two equal Parts.

* 1 *of this.* Upon it make* an Equilateral Triangle ABC, and
† 9 *of this.* biſect † the Angle ACB by the Right Line CD. I ſay, the Right Line AB is biſected in the Point D.

For, becauſe AC is equal to CB, and CD is common, the Right Lines AC, CD, are each equal to the two Right Lines BC, CD, and the Angle ACD
‡ 4 *of this.* equal to the Angle BCD; therefore ‡ the Baſe AD, is equal to the Baſe DB. And ſo the Right Line AB is biſected in the Point D; *which was to be done.*

PROPOSITION XI.

PROBLEM.

To draw a Right Line at Right Angles to a given Right Line, from a given Point in the ſame.

LET AB be the given Right Line, and C the given Point. It is required to draw a Right Line from the Point C at Right Angles to AB.

Aſſume any Point D in AC, and make CE equal * to CD, and upon DE make † the Equilateral Triangle FDE, and join EC. I ſay, the Right Line FC is drawn from the Point C, given in the Right Line AB at Right Angles to AB.

* 3 *of this.*
† 1 *of this.*

For, becauſe DC is equal to CE, and FC is common, the two Lines DC, CF, are each equal to the two Lines EC, CF; and the Baſe DF is equal to the Baſe FE. Therefore * the Angle DCF is equal to the Angle ECF; and they are adjacent Angles. But when a Right Line, ſtanding upon a Right Line, makes the adjacent Angles equal, each of the equal Angles is ‡ a Right Angle; and conſequently DCF, FCE, are both Right Angles. Therefore the Right Line FC, *&c. which was to be done.*

* 8 *of this.*
‡ *Def.* 10.

PROPOSITION XII.

PROBLEM.

To draw a Right Line perpendicular, upon a given infinite Right Line, from a Point given out of it.

LET AB be the given infinite Line, and C the Point given out of it. It is requir'd to draw a Right Line perpendicular upon the given Right Line AB, from the Point C given out of it.

Aſſume any Point D on the other Side of the Right Line AB; and about the Centre C, with the Diſtance CD, deſcribe * a Circle EDG, biſect † EG in H, and join CG, CH, CE. I ſay, there is drawn the Per-

* *Poſt.* 3.
† 10 *of this.*

Perpendicular CH on the given infinite Right Line AB, from the Point C given out of it.

For, becauſe GH is equal to HE, and HC is common, GH and HC are each equal to EH and HC, and the Baſe CG is equal to the Baſe CE. Therefore ‡ 8 *of this.* the Angle CHG is equal ‡ to the Angle CHE; and they are adjacent Angles. But when a Right Line, ſtanding upon another Right Line, makes the Angles equal between themſelves, each of the equal Angles * *Def.* 10. is a Right one*, and the ſaid ſtanding Right Line is call'd a Perpendicular to that which it ſtands on. Therefore CH is drawn *perpendicular, upon a given infinite Right Line, from a given Point out of it*; which was to be demonſtrated.

PROPOSITION XIII.

THEOREM.

When a Right Line, ſtanding upon a Right Line, makes Angles, theſe ſhall be either two Right Angles, or together equal to two Right Angles.

FOR let a Right Line AB, ſtanding upon the Right Line CD, make the Angles CBA, ABD. I ſay, the Angles CBA, ABD, are either two Right Angles, or both together equal to two Right Angles.

* *Def.* 8. For if CBA be equal to ABD, they are * each of † 11 *of this.* them Right Angles: But if not, draw † BE from the Point B, at Right Angles to CD. Therefore the Angles CBE, EBD, are two Right Angles: And becauſe CBE is equal to both the Angles CBA, ABE, add the Angle EBD, which is common; and the two ‡ *Ax.* 2. Angles CBE, EBD, together, are ‡ equal to the three Angles CBA, ABE, EBD, together. Again, becauſe the Angle DBA is equal to the two Angles DBE, EBA, together, add the common Angle ABC, and the two Angles DBA, ABC, are equal to the three Angles DBE, EBA, ABC, together. But it has been prov'd, that the two Angles CBE, EBD, together, are likewiſe equal to theſe three Angles: But Things that are equal to one and the ſame, * *Ax.* 1. are * equal between themſelves. Therefore likewiſe the Angles CBE, EBD, together, are equal to the

Angles DBA, ABC, together; but CBE, EBD, are two Right Angles. Therefore the Angles DBA, ABC, are both together equal to two Right Angles. Wherefore *when a Right Line, ſtanding upon another Right Line, makes Angles, theſe ſhall be either two Right Angles, or together equal to two Right Angles*; which was to be demonſtrated.

PROPOSITION XIV.

THEOREM.

If to any Right Line, and Point therein, two Right Lines be drawn from contrary Parts, making the adjacent Angles, both together, equal to two Right Angles, the ſaid two Right Lines will make but one ſtrait Line.

FOR let two Right Lines BC, BD, drawn from contrary Parts to the Point B, in any Right Line AB, make the adjacent Angles ABC, ABD, both together, equal to two Right Angles. I ſay, BC, BD, make but one Right Line.

For if BD, CB, do not make one ſtrait Line, let CB and BE make one.

Then, becauſe the Right Line AB ſtands upon the Right Line CBE, the Angles ABC, ABE, together, will be equal * to two Right Angles. But the Angles, ABC, ABD, together, are alſo equal to two Right Angles. Now taking away the common Angle ABC, and the remaining Angle ABE is equal to the remaining Angle ABD, the leſs to the greater, which is impoſſible. Therefore BE, BC, are not one ſtrait Line. And in the ſame manner it is demonſtrated, that no other Line but BD is in a ſtrait Line with CB; wherefore CB, BD, ſhall be in one ſtrait Line. Therefore, *if to any Right Line, and Point therein, two Right Lines be drawn from contrary Parts, making the adjacent Angles, both together, equal to two Right Angles, the ſaid two Right Lines will make but one ſtrait Line*; which was to be demonſtrated.

* 13 *of this.*

PRO-

PROPOSITION XV.

THEOREM.

If two Right Lines mutually cut each other, the oppoſite Angles are equal.

LET the two Right Lines AB, CD, mutually cut each other in the Point E. I ſay, the Angle AEC is equal to the Angle DEB; and the Angle CEB equal to the Angle AED.

For, becauſe the Right Line AE, ſtanding on the Right Line CD, makes the Angles CEA, AED:
* 13 *of this.* Theſe both together ſhall be equal * to two Right Angles. Again, becauſe the Right Line DE ſtanding upon the Right Line AB, makes the Angles AED, DEB: Theſe Angles together are *equal to two Right Angles. But it has been prov'd, that the Angles CEA, AED, are likewiſe together equal to two Right Angles. Therefore the Angles CEA, AED, are equal to the Angles AED, DEB. Take away the common
† *Ax.* 3. Angle AED, and the Angle remaining CEA, is † equal to the Angle remaining BED. For the ſame Reaſon, the Angle CEB ſhall be equal to the Angle DEA. Therefore, *if two Right Lines mutually cut each other, the oppoſite Angles are equal*; which was to be demonſtrated.

Coroll. 1. From hence it is manifeſt, that two Right Lines mutually cutting each other, make Angles at the Section equal to four Right Angles.

Coroll. 2. All the Angles conſtituted about the ſame Point, are equal to four Right Angles.

PROPOSITION XVI.

THEOREM.

If one Side of any Triangle be produced, the outward Angle is greater than either of the inward oppoſite Angles.

LET ABC be a Triangle, and one of its Sides BC, be produced to D. I ſay, the outward Angle ACD is greater than either of the inward Angles CBA, or BAC.

For biſect AC in E*, and join BE, which produce to F, and make EF equal to BE. Moreover, join FC, and produce AC to G.

* 10 *of this.*

Then, becauſe AE is equal to EC, and BE to EF, the two Sides AE, EB, are equal to the two Sides CE, EF, each to each, and the Angle AEB † equal to the Angle FEC; for they are oppoſite Angles. Therefore the Baſe AB is ‡ equal to the Baſe FC; and the Triangle AEB, equal to the Triangle FEC; and the remaining Angles of the one, equal to the remaining Angles of the other, each to each, ſubtending the equal Sides. Wherefore the Angle BAE is equal to the Angle ECF; but the Angle ECD is greater than the Angle ECF; therefore the Angle ACD, is greater than the Angle BAE. After the ſame manner, if the Right Line BC be biſected, we demonſtrate that the Angle BCG, that is, the Angle ACD, is greater than the Angle ABC. Therefore, *one Side of any Triangle being produced, the outward Angle is greater than either of the inward oppoſite Angles*; which was to be demonſtrated.

† 15 *of this.*
‡ 4 *of this.*

PROPOSITION XVII.

THEOREM.

Two Angles of any Triangle together, howſoever taken, are leſs than two Right Angles.

LET ABC be a Triangle. I ſay, two Angles of it together, howſoever taken, are leſs than two Right Angles.

For produce BC to D.

Then, becauſe the outward Angle ACD of the Triangle ABC is greater * than the inward oppoſite Angle ABC: If the common Angle ACB be added, the Angles ACD, ACB, together, will be greater than the Angles ABC, ACB together: But ACD, ACB are † equal to two Right Angles. Therefore ABC, BCA, are leſs than two Right Angles. In the ſame manner we demonſtrate, that the Angles BAC, ACB, as alſo CAB, ABC, are leſs than two Right Angles. Therefore *two Angles of any Triangle together, howſoever taken, are leſs than two Right Angles*; which was to be demonſtrated.

* 16 *of this.*

† 13 *of this.*

PROPOSITION XVIII.

THEOREM.

The greater Side of every Triangle ſubtends the greater Angle.

LET ABC be a Triangle, having the Side AC greater than the Side AB. I ſay, the Angle ABC is greater than the Angle BCA.

For, becauſe AC is greater than AB, AD may be made equal to AB, and BD be joined.

Then, becauſe ADB is an outward Angle of the Triangle BDC, it will be * greater than the inward oppoſite Angle DCB. But ADB is † equal to ABD; becauſe the Side AB is equal to the Side AD. Therefore the Angle ABD is likewiſe greater than the Angle ACB; and conſequently ABC ſhall be much greater than ACB. Wherefore *the greater Side of every Triangle ſubtends the greater Angle*; which was to be demonſtrated.

* 16 *of this.*

† 5 *of this.*

PRO-

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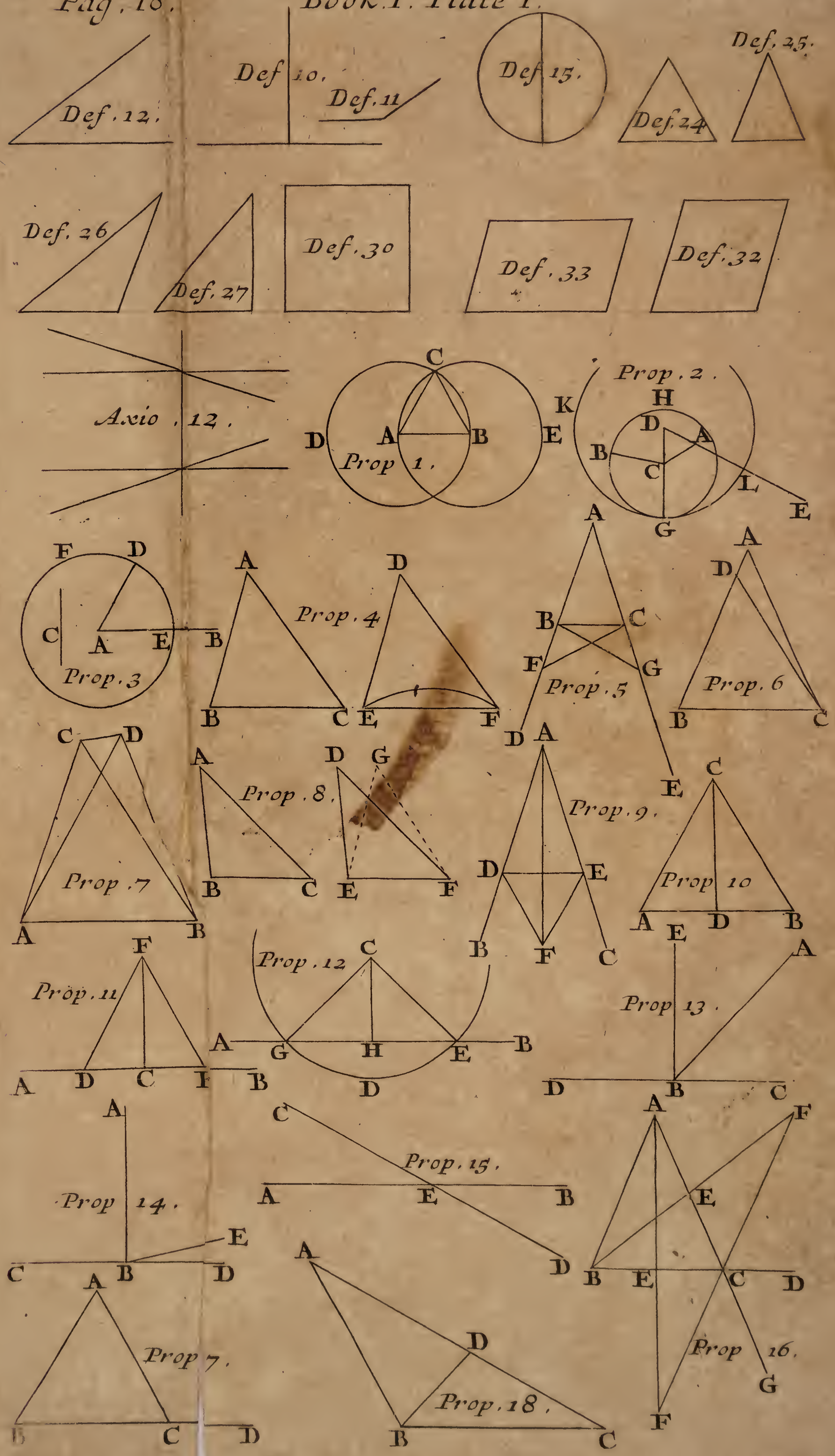


PROPOSITION XIX.

THEOREM.

The greater Angle of every Triangle ſubtends the greater Side.

LET ABC be a Triangle, having the Angle ABC greater than the Angle BCA. I ſay, the Side AC is greater than the Side AB.

For, if it be not greater, AC is either equal to AB, or leſs than it. It is not equal to it, becauſe then the Angle ABC would be equal *to the Angle ACB; but it is not: Therefore AC is not equal to AB. Neither will it be leſs; for then the Angle ABC would be † leſs than the Angle ACB; but it is not. Therefore AC is not leſs than AB. But likewiſe it has been proved not to be equal to it: Wherefore AC is greater than AB. Therefore *the greater Angle of every Triangle ſubtends the greater Side*; which was to be demonſtrated.

* 5 *of this.*

† 18 *of this.*

PROPOSITION XX.

THEOREM.

Two Sides of any Triangle, howſoever taken, are together greater than the third Side.

LET ABC be a Triangle: I ſay, two Sides thereof, howſoever taken, are together greater than the third Side, *viz.* the Sides BA, AC, are greater than the Side BC; and the Sides AB, BC, greater than the Side AC; and the Sides BC, CA, greater than the Side AB.

For produce BA to the Point D, ſo that AD be * equal to AC, and join DC.

Then, becauſe DA is equal to AC, the Angle ADC ſhall be equal † to the Angle ACD. But the Angle BCD is greater than the Angle ACD. Wherefore the Angle BCD is greater than the Angle ADC; and becauſe DCB is a Triangle, having the Angle BCD greater than the Angle BDC, and the greater

* 3 *of this.*

† 5 *of this.*

* 19 *of this.* Angle ſubtends * the greater Side; the Side D B will be greater than the Side B C. But D B is equal to B A and A C together. Wherefore the Sides B A, A C, together, are greater than the Side B C. In the ſame manner we demonſtrate, that the Sides A B, B C, together, are greater than the Side C A; and the Sides B C, C A, together, are greater than the Side A B. Therefore *two Sides of any Triangle, howſoever taken, are together greater than the third Side*; which was to be demonſtrated.

PROPOSITION XXI.

THEOREM.

If two Right Lines be drawn from the extreme Points of one Side of a Triangle to any Point within the ſame, theſe two Lines ſhall be leſs than the other two Sides of the Triangle, but contain a greater Angle.

FOR let two Right Lines B D, D C, be drawn from the Extremes B, C, of the Side B C of the Triangle A B C, to the Point D within the ſame. I ſay, B D, D C, are leſs than B A, A C, the other two Sides of the Triangle, but contain an Angle B D C greater than the Angle B A C.

For produce B D to E.

Then, becauſe two Sides of every Triangle together * 20 *of this.* are * greater than the third, B A, A E, the two Sides of the Triangle A B E, are greater than the Side B E. Now, add E C, which is common, and the Sides † *Ax.* 4. B A, A C, will be † greater than B E, E C.

Again, becauſe C E, E D, the two Sides of the Triangle C E D, are greater than the Side C D, add D B, which is common, and the Sides C E, E B, will be greater than C D, D B. But it has been proved, that B A, A C, are greater than B E, E C: Wherefore B A, A C, are much greater than B D, D C. Again, becauſe ‡ 16 *of this.* the outward Angle of every Triangle is ‡ greater than the inward and oppoſite one: B D C, the outward Angle of the Triangle C D E, ſhall be greater than the Angle C E D. For the ſame Reaſon, C E B, the outward Angle of the Triangle A B E, is likewiſe greater than the Angle B A C; but the Angle B D C has been proved to be greater than the Angle C E B. Wherefore the Angle B D C ſhall be much greater than the Angle B A C. And ſo, *if two Right Lines be drawn from the extreme Points of one Side of a Triangle to any Point within the ſame, theſe two Lines ſhall be leſs than the other two Sides of the Triangle, but contain a greater Angle*; which was to be demonſtrated.

PROPOSITION XXII.

PROBLEM.

To deſcribe a Triangle of three Right Lines, which are equal to three others given: But it is requiſite, that any two of the Right Lines taken together be greater than the third; becauſe two Sides of a Triangle, howſoever taken, are together greater than the third Side.

LET A, B, C, be three Right Lines given, two of which, any ways taken, are greater than the third, *viz.* A and B together greater than C; A and C greater than B; and B and C greater than A. Now it is required to make a Triangle of three Right Lines equal to A, B, C: Let there be one Right Line D E, terminated at D, but infinite towards E; and take * D F equal to A, F G equal to B, and G H equal to C; and about the Centre F, with the Diſtance F D, deſcribe a Circle D K L †; and about the Centre G, with the Diſtance G H, deſcribe another Circle K L H, and join K F, K G. I ſay, the Triangle K F G is made of three Right Lines, equal to A, B, C; for, becauſe the Point F is the Centre of the Circle D K, F K ſhall be equal to F D: But F D is equal to A; therefore F K is alſo equal to A. Again, becauſe the Point G is the Centre of the Circle L K H, G K will be ‡ equal to G H: But G H is equal to C; therefore ſhall G K be alſo equal to C: But F G is likewiſe equal to B; and conſequently the three Right Lines K F, F G, K G, are equal to the three Right Lines A, B, C; wherefore the Triangle K F G is made of three Right Lines K F, F G, G K, equal to the three given Lines A, B, C; *which was to be done.*

* *3 of this.*

† *3 Poſt.*

‡ *Def. 15.*

PROPOSITION XXIII.

PROBLEM.

With a given Right Line, and at a given Point in it, to make a Right-lin'd Angle equal to a Right-lin'd Angle given.

LET the given Right Line be AB, and the Point given therein A, and the given Right-lin'd Angle DCE. It is required to make a Right-lin'd Angle at the given Point A, with the given Right Line AB, equal to the given Right-lin'd Angle DCE.

Aſſume the Points D and E at Pleaſure in the Lines CD, CE, and draw DE; then, of three Right Lines equal to CD, DE, EC, make *a Triangle AFG, ſo that AF be equal to CD, AG to CE, and FG to DE. * 22 *of this.*

Then, becauſe the two Sides DC, CE, are equal to the two Sides FA, AG, each to each, and the Baſe DE equal to the Baſe FG; the Angle DCE ſhall be †equal to the Angle FAG. Therefore the Right-lined Angle FAG is made at the given Point A, in the given Line AB, equal to the given Right-lined Angle DCE; *which was to be done.* † 8 *of this.*

PROPOSITION XXIV.

THEOREM.

If two Triangles have two Sides of the one, equal to two Sides of the other, each to each, and the Angle of the one, contained under the equal Right Lines, greater than the correſpondent Angle of the other; then the Baſe of the one will be greater than the Baſe of the other.

LET there be two Triangles ABC, DEF, having two Sides AB, AC, equal to the two Sides DE, DF, each to each, *viz.* the Side AB equal to the Side DE, and the Side AC equal to DF; and let the Angle BAC be greater than the Angle EDF. I ſay, the Baſe BC is greater than the Baſe EF.

For, becauſe the Angle B A C is greater than the Angle E D F, make * an Angle E D G at the Point D in the Right Line D E, equal to the Angle B A C, and make † D G equal to either A C or D F, and join E F, F G.

* 23 *of this.*
† 3 *of this.*

Now, becauſe A B is equal to D E, and A C to D G, the two Sides B A, A C, are each equal to the two Sides E D, D G, and the Angle B A C equal to the Angle E D G: Therefore the Baſe B C is equal ‡ to the Baſe E G. Again, becauſe D G is equal to D F, the Angle D F G is ‡ equal to the Angle D G F; and ſo the Angle D F G is greater than the Angle E G F: And conſequently the Angle E F G is much greater than the Angle E G F. And becauſe E F G is a Triangle, having the Angle E F G greater than the Angle E G F; and the greateſt Side ſubtends ‖ the greateſt Angle, the Side E G ſhall be greater than the Side E F. But the Side E G is equal to the Side B C: Whence B C is likewiſe greater than E F. Therefore, *if two Triangles have two Sides of the one, equal to two Sides of the other, each to each, and the Angle of the one, contained under the equal Right Lines, greater than the correſpondent Angle of the other; then the Baſe of the one will be greater than the Baſe of the other;* which was to be demonſtrated.

‡ 4 *of this.*
‡ 5 *of this.*
‖ 19 *of this.*

PROPOSITION XXV.

THEOREM.

If two Triangles have two Sides of the one equal to two Sides of the other, each to each, and the Baſe of the one greater than the Baſe of the other; they ſhall alſo have the Angles contained under the equal Sides, the one greater than the other.

LET there be two Triangles A B C, D E F, having two Sides A B, A C, each equal to two Sides D E, D F, *viz.* the Side A B equal to the Side D E, and the Side A C to the Side D F; but the Baſe B C greater than the Baſe E F. I ſay, the Angle B A C is alſo greater than the Angle E D F.

For if it be not greater, it will be either equal or leſs. But the Angle B A C is not equal to the Angle

* 4 *of this.* EDF; for if it was, the Baſe BC would be * equal to the Baſe EF; but it is not: Therefore the Angle BAC is not equal to the Angle EDF, neither will it † 24 *of this.* be leſſer; for if it ſhould, the Baſe BC would be † leſs than the Baſe EF; but it is not. Therefore the Angle BAC is not leſs than the Angle EDF; but it has likewiſe been proved not to be equal to it. Wherefore the Angle BAC is neceſſarily greater than the Angle EDF. *If,* therefore, *two Triangles have two Sides of the one equal to two Sides of the other, each to each, and the Baſe of the one greater than the Baſe of the other; they ſhall alſo have the Angles, contained under the equal Sides, the one greater than the other;* which was to be demonſtrated.

PROPOSITION XXVI.

THEOREM.

If two Triangles have two Angles of the one equal to two Angles of the other, each to each, and one Side of the one equal to one Side of the other, either the Side lying between the equal Angles, or which ſubtends one of the equal Angles; the remaining Sides of the one Triangle ſhall be alſo equal to the remaining Sides of the other, each to his correſpondent Side, and the remaining Angle of the one, equal to the remaining Angle of the other.

LET there be two Triangles ABC, DEF, having two Angles ABC, BCA of the one, equal to two Angles DEF, EFD, of the other, each to each, that is, the Angle ABC equal to the Angle DEF, and the Angle BCA equal to the Angle EFD. And let one Side of the one be equal to one Side of the other, which firſt let be the Side lying between the equal Angles, *viz.* the Side BC equal to the Side EF. I ſay, the remaining Sides of the one Triangle will be equal to the remaining Sides of the other, each to each, that is, the Side AB equal to the Side DE, and the Side AC equal to the Side DF, and the remaining Angle BAC equal to the remaining Angle EDF.

For if the Side AB be not equal to the Side DE, one of them will be the greater, which let be AB, make GB equal to DE, and join GC.

Then, becauſe BG is equal to DE, and BC to EF, the two Sides GB, BC, are equal to the two Sides DE, EF, each to each; and the Angle GBC equal to the Angle DEF. The Baſe GC is * equal to the * 4 *of this.* Baſe DF, and the Triangle GBC to the Triangle DEF, and the remaining Angles equal to the remaining Angles, each to each, which ſubtend the equal Sides. Therefore the Angle GCB is equal to the Angle DFE. But the Angle DFE, by the Hypotheſis, is equal to the Angle BCA; and ſo the Angle BCG is likewiſe equal to the Angle BCA, the leſs to the greater, which cannot be. Therefore AB is not unequal to DE, and conſequently is equal to it. And ſo the two Sides AB, BC, are each equal to the two Sides DE, EF, and the Angle ABC equal to the Angle DEF: And conſequently the Baſe AC * is equal to the Baſe DF, and the remaining Angle BAC equal to the remaining Angle EDF.

Secondly, Let the Sides that are ſubtended by the equal Angles be equal, as AB equal to DE. I ſay, the remaining Sides of the one Triangle are equal to the remaining Sides of the other, *viz.* AC to DF, and BC to EF; and alſo the remaining Angle BAC, to the remaining Angle EDF.

For if BC be unequal to EF, one of them is the greater, which let be BC, if poſſible, and make BH equal to EF, and join AH.

Now, becauſe BH is equal to EF, and AB to DE, the two Sides AB, BH, are equal to the two Sides DE, EF, each to each, and they contain equal Angles: Therefore the Baſe AH is * equal to the Baſe DF; and the Triangle ABH ſhall be equal to the Triangle DEF, and the remaining Angles equal to the remaining Angles, each to each, which ſubtend the equal Sides: And ſo the Angle BHA is equal to the Angle EFD. But EFD is † equal to the Angle † *From the Hyp.* BCA; and conſequently the Angle BHA is equal to the Angle BCA: Therefore the outward Angle BHA of the Triangle AHC, is equal to the inward and oppoſite Angle BCA; which is ‡ impoſſible: ‡ 16 *of this.* Whence BC is not unequal to EF; therefore it is equal to it. But AB is alſo equal to DE. Wherefore the two Sides AB, BC, are equal to the two Sides DE, EF, each to each; and they contain equal Angles. And ſo the Baſe AC is equal to the Baſe DF, the Triangle BAC to the Triangle DEF, and the remaining Angle BAC equal to the remaining Angle EDF. *If*, therefore, *two Triangles have two Angles equal, each to each, and one Side of the one equal to one Side of the other, either the Side lying between the equal Angles, or which ſubtends one of the equal Angles; the remaining Sides of the one Triangle ſhall be alſo equal to the remaining Sides of the other, each to his correſpondent Side, and the remaining Angle of the one equal to the remaining Angle of the other*; which was to be demonſtrated.

PROPOSITION XXVII.

THEOREM.

If a Right Line, falling upon two Right Lines, makes the alternate Angles equal between themſelves, the two Right Lines ſhall be parallel.

LET the Right Line EF, falling upon two Right Lines AB, CD, make the alternate Angles AEF, EFD, equal between themſelves. I ſay, the Right Line AB is parallel to CD.

For if it be not parallel, AB and CD, produced towards B and D, or towards A and C, will meet: Now let them be produced towards B and D, and meet in the Point G.

Then the outward Angle AEF of the Triangle GEF, is * greater than the inward and oppoſite Angle EFG, and alſo equal † to it; which is abſurd. Therefore AB and CD, produced towards B and D, will not meet each other. By the ſame way of Reaſoning, neither will they meet, being produced towards C and A. But Lines that meet each other on neither Side, are ‡ parallel between themſelves. Therefore AB is parallel to CD. Therefore, *if a Right Line, falling upon two Right Lines, makes the alternate Angles equal between themſelves, the two Right Lines ſhall be parallel*; which was to be demonſtrated.

* 16 *of this.*
† *From the Hyp.*
‡ *Def.* 35.

PRO-

PROPOSITION XXVIII.

THEOREM.

If a Right Line, falling upon two Right Lines, makes the outward Angle of the one Line equal to the inward and oppoſite Angle of the other on the ſame Side, or the inward Angles on the ſame Side together equal two Right Angles, the two Right Lines ſhall be parallel between themſelves.

LET the Right Line EF, falling upon two Right Lines AB, CD, make the outward Angle EGB equal to the inward and oppoſite Angle GHD; or the inward Angles BGH, GHD on the ſame Side together equal to two Right Angles. I ſay, the Right Line AB is parallel to the Right Line CD.

For, becauſe the Angle EGB is *equal to the Angle GHD, and the Angle EGB † equal to the Angle AGH, the Angle AGH ſhalll be equal to the Angle GHD; but theſe are alternate Angles. Therefore AB is ‡ parallel to CD.

Again, becauſe the Angles BGH, GHD, are equal to two Right Angles, and AGH, BGH, are *equal to two Right ones, the Angles AGH, BGH, will be equal to the Angles BGH, GHD; and if the common Angle BGH be taken from both, there will remain the Angle AGH equal to the Angle GHD; but theſe are alternate Angles. Therefore AB is parallel to CD. *If*, therefore, *a Right Line, falling upon two Right Lines, makes the outward Angle of the one Line equal to the inward and oppoſite Angle of the other on the ſame Side, or the inward Angles on the ſame Side together equal to two Right Angles, the two Right Lines ſhall be parallel between themſelves;* which was to be demonſtrated.

* *From the Hyp.*

† 15 *of this.*

‡ 27 *of this.*

* 13 *of this.*

PRO-

PROPOSITION XXIX.

THEOREM.

If a Right Line falls upon two Parallels, it will make the alternate Angles equal between themſelves; the outward Angle equal to the inward and oppoſite Angle, on the ſame Side; and the inward Angles on the ſame Side together equal to two Right Angles.

LET the Right Line EF fall upon the parallel Right Lines AB, CD. I ſay, the alternate Angles AGH, GHD, are equal between themſelves; the outward Angle EGB is equal to the inward one GHD, on the ſame Side; and the two inward ones, BGH, GHD, on the ſame Side, are together equal to two Right Angles.

For if AGH be unequal to GHD, one of them will be the greater. Let this be AGH; then, becauſe the Angle AGH is greater than the Angle GHD, add the common Angle BGH to both: And ſo the Angles AGH, BGH, together, are greater than the Angles BGH, GHD, together. But the Angles * 13 *of this.* AGH, BGH, are equal to two Right ones*. Therefore BGH, GHD, are leſs than two Right Angles. † *Ax.* 12. And ſo the Lines AB, CD, infinitely produced †, will meet each other; but becauſe they are parallel, they will not meet. Therefore the Angle AGH is not unequal to the Angle GHD. Wherefore it is neceſſarily equal to it.

‡ 15 *of this.* But the Angle AGH is ‡ equal to the Angle EGB: Therefore EGB is alſo equal to GHD.

Now add the common Angle BGH, and then EGB, BGH, together, are equal to BGH, GHD, together; but EGB, and BGH, are equal to two Right Angles. Therefore alſo BGH, and GHD, ſhall be equal to two Right Angles. Wherefore, *if a Right Line falls upon two Parallels, it will make the alternate Angles equal between themſelves; the outward Angle equal to the inward and oppoſite Angle, on the ſame Side, and the inward Angles on the ſame Side together equal to two Right Angles*; which was to be demonſtrated.

PROPOSITION XXX.

THEOREM.

Right Lines parallel to one and the ſame Right Line, are alſo parallel between themſelves.

LET AB and CD be Right Lines, each of which is parallel to the Right Line EF. I ſay, AB is alſo parallel to CD. For let the Right Line GK fall upon them.

Then, becauſe the Right Line GK falls upon the parallel Right Lines AB, EF, the Angle AGH is*equal to the Angle GHF; and becauſe the Right Line GK, falls upon the parallel Right Lines EF, CD, the Angle GHF is equal to the Angle GKD*. But it has been proved, that the Angle AGK is alſo equal to the Angle GHF. Therefore AGK is equal to GKD, and they are alternate Angles; whence AB is parallel to CD†. And ſo *Right Lines parallel to one and the ſame Right Line, are parallel between themſelves*; which was to be demonſtrated.

* 29 *of this.*

† 27 *of this.*

PROPOSITION XXXI.

PROBLEM.

To draw a Right Line thro' a given Point parallel to a given Right Line.

LET A be a Point given, and BC a Right Line given. It is required to draw a Right Line thro' the Point A, parallel to the Right Line BC.

Aſſume any Point D in BC, and join AD; then make *an Angle DAE, at the Point A, with the Line DA, equal to the Angle ADC, and produce EA ſtrait forwards to F.

Then, becauſe the Right Line AD, falling on two Right Lines BC, EF, makes the alternate Angles EAD, ADC, equal between themſelves, EF ſhall be †parallel to BC. Therefore the Right Line EAF is drawn thro' the given Point A, parallel to the given Right Line BC; *which was to be done.*

* 23 *of this.*

† 27 *of this.*

* *Coroll.* Hence it appears, that if one Angle of any Triangle be equal to the other two, that is a Right one; becauſe that the Angle adjacent to this Right one, is equal to the other two. But when adjacent Angles are equal, they are neceſſarily Right ones.

PROPOSITION XXXII.

THEOREM.

If one Side of any Triangle be produced, the outward Angle is equal to both the inward and oppoſite Angles; and the three inward Angles of a Triangle are equal to two Right Angles.

LET ABC be a Triangle, one of whoſe Sides BC is produced to D. I ſay, the outward Angle ACD is equal to the two inward and oppoſite Angles CAB, ABC; and the three inward Angles of the Triangle, *viz.* ABC, BCA, CAB, are equal to two Right Angles.

* 31 *of this.* For let CE be drawn * thro' the Point C, parallel to the Right Line AB. Then, becauſe AB is parallel to CE, and AC falls upon them, the alternate Angles † 29 *of this.* BAC, ACE, are † equal between themſelves. Again, becauſe AB is parallel to CE, and the Right Line BD falls upon them, the outward Angle ECD is † equal to the inward and oppoſite one ABC; but it has been proved, that the Angle ACE is equal to the Angle BAC. Wherefore the whole outward Angle ACD is equal to both the inward and oppoſite Angles BAC, ABC. And if the Angle ACB, which is common, be added, the two Angles ACD, ACB, are equal to the three Angles ABC, BAC, ‡ 13 *of this.* ACB; but the Angles ACD, ACB, are ‡ equal to two Right Angles. Therefore alſo ſhall the Angles ACB, CBA, CAB, be equal to two Right Angles. Wherefore, *if one Side of any Triangle be produced, the outward Angle is equal to both the inward and oppoſite Angles, and the three inward Angles of a Triangle are equal to two Right Angles*; which was to be demonſtrated.

Coroll. 1. All the three Angles of any one Triangle taken together, are equal to all the three Angles of any other Triangle taken together.

Coroll. 2. If two Angles of any one Triangle, either ſeparately or taken together, be equal to two Angles of any other Triangle; then the remaining Angle of the one Triangle, will be equal to the remaining Angle of the other.

Coroll. 3. If one Angle of a Triangle be a Right Angle, the other two Angles together make one Right Angle.

Coroll. 4. If the Angle included between the equal Legs of an Iſoſceles Triangle be a Right one, each of the other Angles at the Baſe will be half Right Angles.

Coroll. 5. Any Angle in an Equilateral Triangle is equal to one Third of two Right Angles, or two Thirds of one Right Angle.

THEOREM I.

All the inward Angles of any Right-lin'd Figure whatſoever, make twice as many Right Angles, abating four, as the Figure has Sides.

FOR any Right-lin'd Figure may be reſolved into as many Triangles, abating two, as it hath Sides. For Example, if a Figure has four Sides, it may be reſolved into two Triangles: If a Figure has five Sides, it may be reſolved into three Triangles; if ſix, into four; and ſo on. Wherefore (by Prop. XXXII.) *the Angles of all theſe Triangles are equal to twice as many Right Angles as there are Triangles: But the Angles of all the Triangles are equal to the inward Angles of the Figure. Therefore all the inward Angles of the Figure are equal to twice as many Right Angles as there are Triangles, that is, twice as many Right Angles, taking away four, as the Figure has Sides.* W. W. D.

THEOREM II.

All the outward Angles of any Right-lin'd Figure together, make four Right Angles.

FOR the outward Angles, together with the inward ones, make twice as many Right Angles as the Figure has Sides; but from the laſt Theorem, all the inward Angles together make twice as many Right Angles, abating four, as the Figure has Sides. Wherefore the outward Angles are all together equal to four Right Angles. W. W. D.

PROPOSITION XXXIII.

THEOREM.

Two Right Lines, which join two equal and parallel Right Lines, towards the ſame Parts, are alſo equal and parallel.

LET the parallel and equal Right Lines AB, CD, be joined towards the ſame Parts, by the Right Lines AC, BD. I ſay, AC, BD, are equal and parallel.

For draw BC.

Then, becauſe AB is parallel to CD, and BC falls upon them, the alternate Angles ABC, BCD, are
* 29 *of this.* * equal. Again, becauſe AB is equal to CD, and BC is common; the two Sides AB, BC, are each equal to the two Sides BC, CD; but the Angle ABC is alſo equal to the Angle BCD; therefore the
† 4 *of this.* Baſe AC is † equal to the Baſe BD: And the Triangle ABC, equal to the Triangle BCD; and the remaining Angles equal to the remaining Angles, each to each, which ſubtend the equal Sides. Wherefore the Angle ACB is equal to the Angle CBD. And becauſe the Right Line BC, falling upon two Right Lines AC,
‡ 27 *of this.* BD, makes ‡ the alternate Angles ACB, CBD, equal between themſelves; AC is ‡ parallel to BD. But it has been proved alſo to be equal to it. Therefore *two Right Lines, which join two unequal and parallel Right Lines, towards the ſame Parts, are alſo equal and parallel*; which was to be demonſtrated.

Defin. *A Parallelogram is a Quadrilateral Figure, each of whoſe oppoſite Sides are parallel.*

PROPOSITION XXXIV.

THEOREM.

The oppoſite Sides and oppoſite Angles of any Parallelogram are equal; and the Diameter divides the ſame into two equal Parts.

LET ABDC be a Parallelogram, whoſe Diameter is BC. I ſay, the oppoſite Sides and oppoſite Angles are equal between themſelves, and the Diameter BC biſects the Parallelogram.

For, becauſe AB is parallel to CD, and the Right Line BC falls on them, the alternate Angles ABC, BCD, are * equal between themſelves; again, becauſe AC is parallel to BD, and BC falls upon them, the alternate Angles ACB and CBD are equal to one another. Wherefore ABC, CBD, are two Triangles, having two Angles ABC, BCA, of the one, equal to two Angles BCD, CBD, of the other, each to each; and likewiſe one Side of the one equal to one Side of the other, *viz.* the Side BC between the equal Angles, which is common. Therefore the remaining Sides ſhall be † equal to the remaining Sides, each to each, and the remaining Angle to the remaining Angle. And ſo the Side AB is equal to the Side CD, the Side AC to BD, and the Angle BAC to the Angle BDC. And becauſe the Angle ABC is equal to the Angle BCD, and the Angle CBD to the Angle ACB; therefore the whole Angle ABD is equal to the whole Angle ACD: But it has been proved, that the Angle BAC is alſo equal to the Angle BDC.

* 29 *of this.*

† 26 *of this.*

Wherefore the oppoſite Sides and Angles of any Parallelogram are equal between themſelves.

I ſay, moreover, that the Diameter biſects it. For becauſe AB is equal to CD, and BC is common, the two Sides AB, BC, are each equal to the two Sides DC, CB; and the Angle ABC is alſo equal to the Angle BCD. Therefore the Baſe AC is ‡ equal to the Baſe DB; and the Triangle ABC is equal

‡ 4 *of this.*

equal to the Triangle BCD. Wherefore the Diameter BC biſects the Parallelogram ACDB; *which was to be demonſtrated.*

PROPOSITION XXXV.

THEOREM.

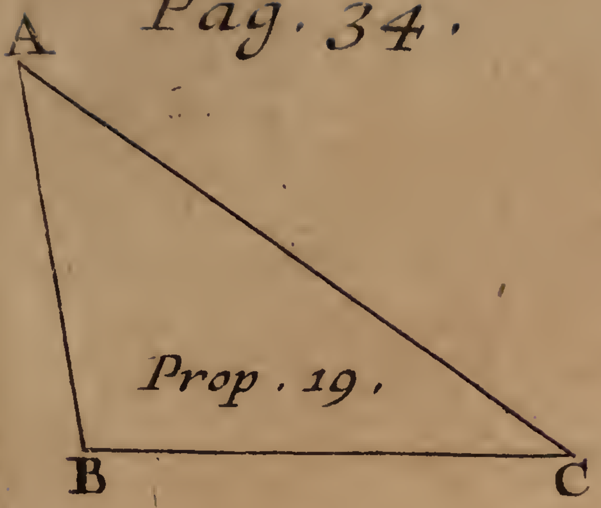
Parallelograms conſtituted upon the ſame Baſe, and between the ſame Parallels, are equal between themſelves.

LET ABCD, EBCF, be Parallelograms conſtituted upon the ſame Baſe BC, and between the ſame Parallels AF and BC. I ſay, the Parallelogram ABCD, is equal to the Parallelogram EBCF.

For becauſe ABCD is a Parallelogram, AD is * equal to BC; and for the ſame Reaſon EF is equal to BC; wherefore AD ſhall be † equal to EF; but DE is common. Therefore the Whole AE is ‡ equal to the Whole DF. But AB is equal to DC; wherefore EA, AB, the two Sides of the Triangle ABE, are equal to the two Sides FD, DC, each to each; and the Angle FDC * equal to the Angle EAB, the outward one to the inward one. Therefore the Baſe EB is † equal to the Baſe CF, and the Triangle EAB to the Triangle FDC. If the common Triangle DGE be taken from both, there will remain ‡ the Trapezium ABGD, equal to the Trapezium FCGE; and if the Triangle GBC, which is common, be added, the Parallelogram ABCD will be equal to the Parallelogram EBCF. Therefore *Parallelograms conſtituted upon the ſame Baſe, and between the ſame Parallels, are equal between themſelves;* which was to be demonſtrated.

* 34 *of this.*
† *Ax.* 1.
‡ *Ax.* 2.
* 29 *of this.*
† 4 *of this.*
‡ *Ax.* 3.

Pag. 34. *Book 1. Plate 2.*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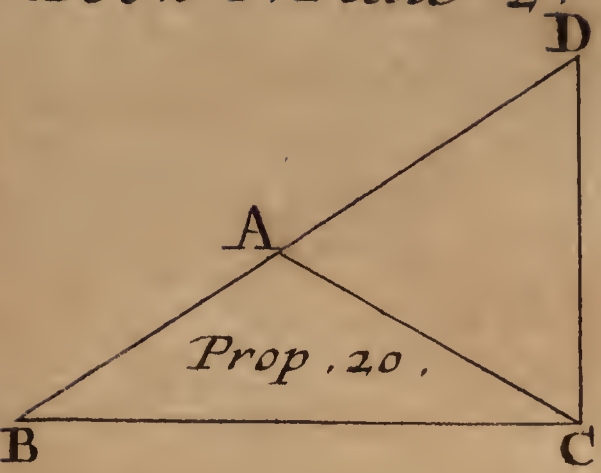
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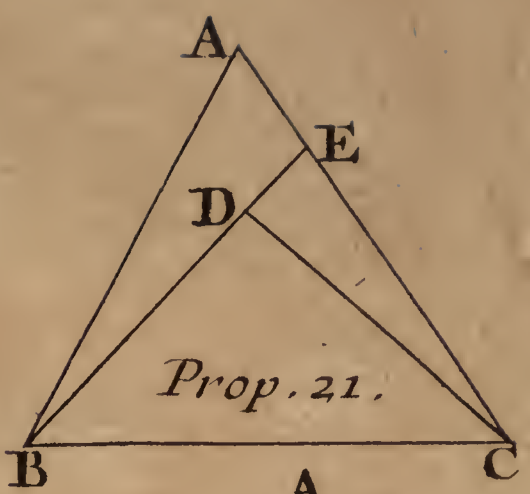
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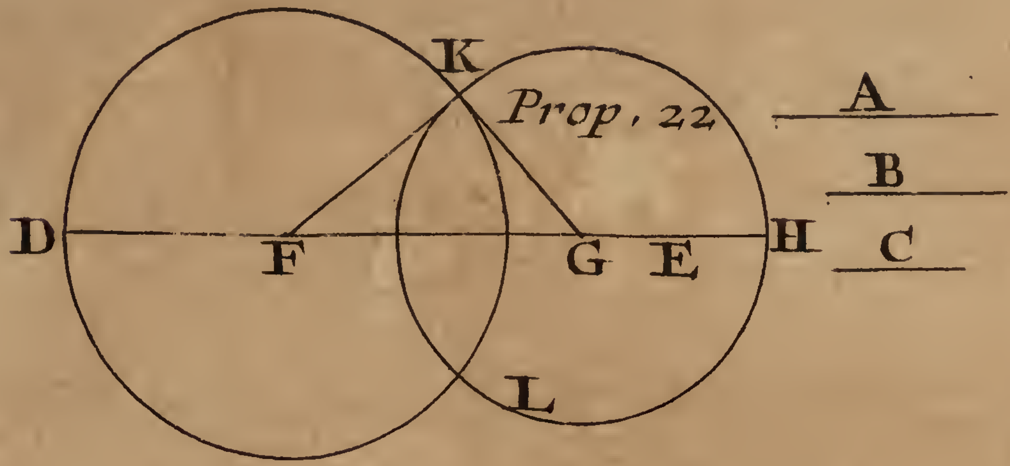
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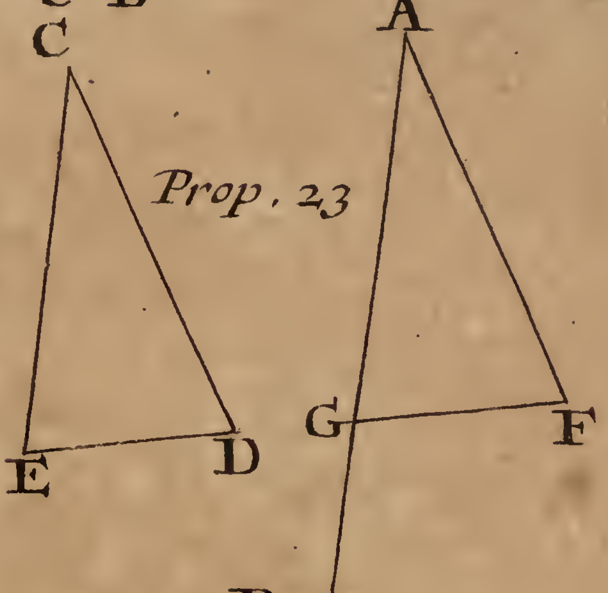
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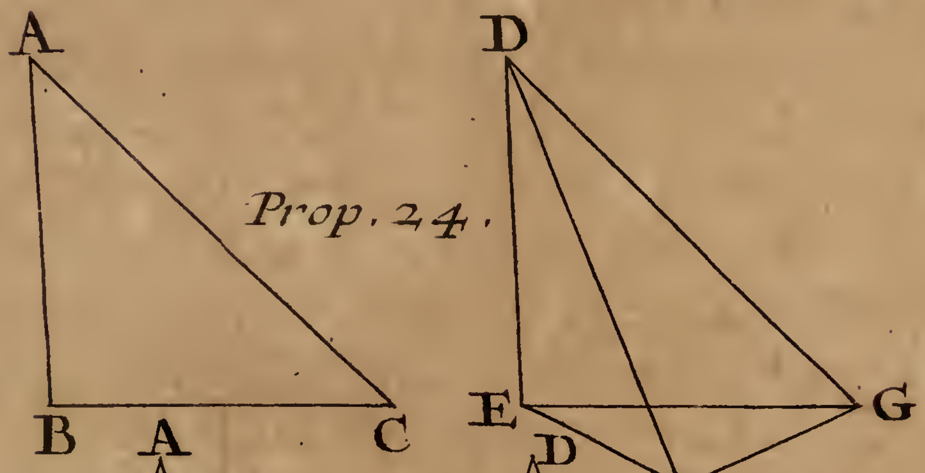
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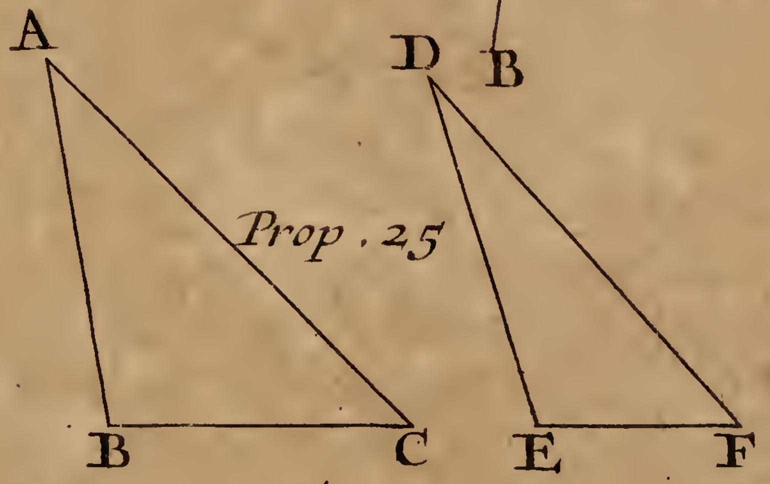
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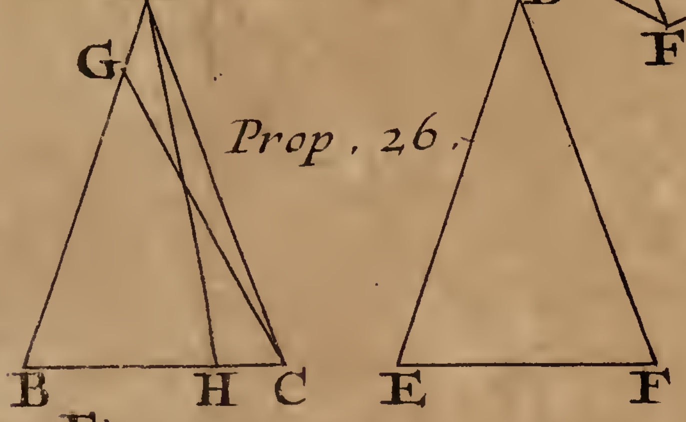
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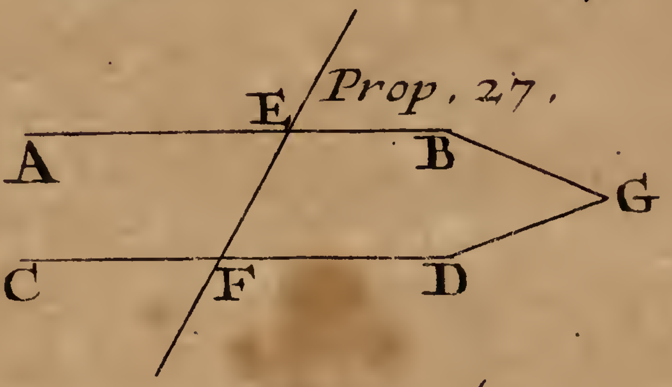
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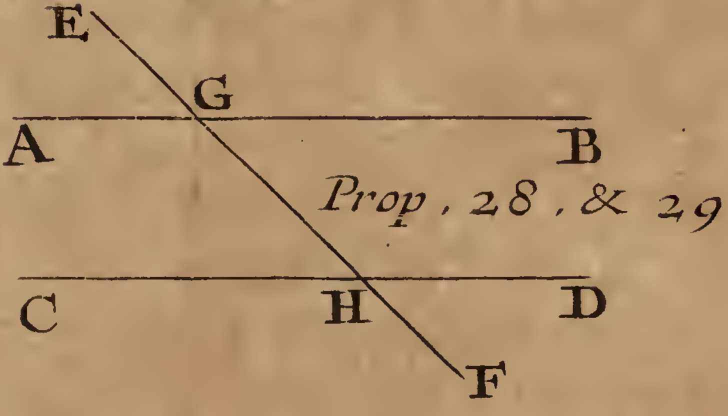
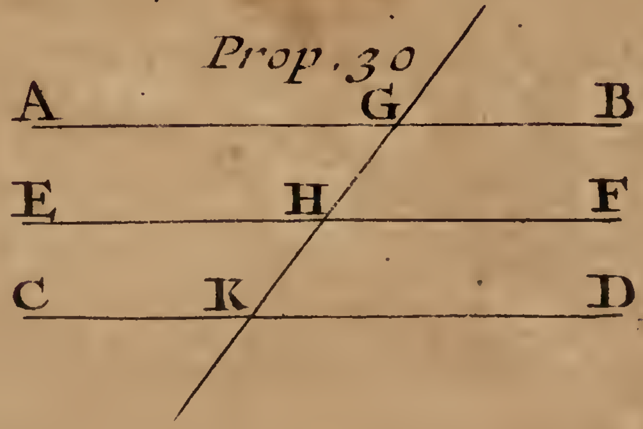

Prop. 28. & 29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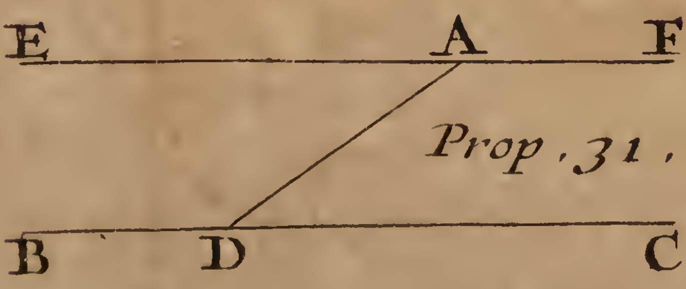
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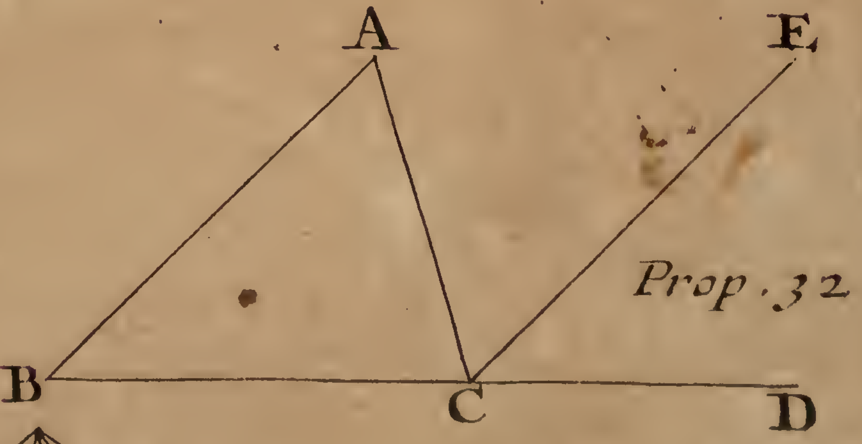
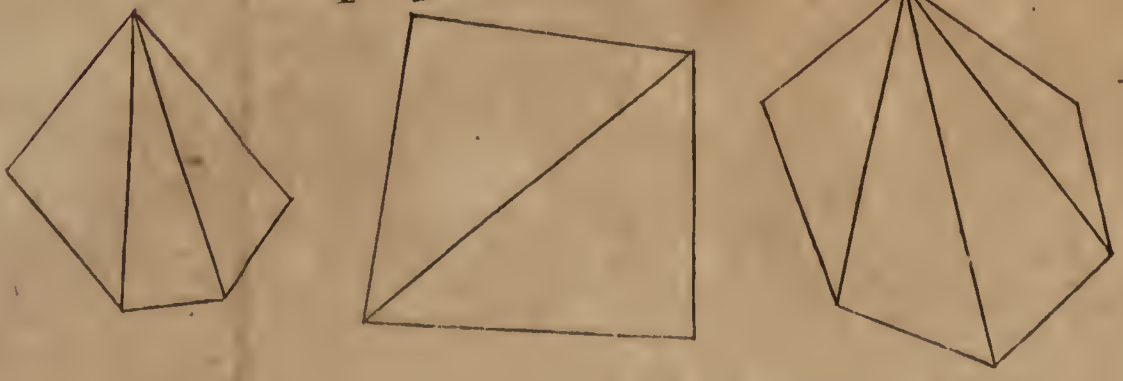
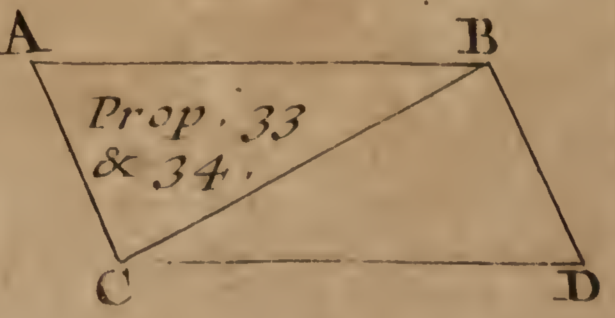

Prop. 32

Prop. 32 Theorem




Prop. 33 & 34.

PROPOSITION XXXVI.

THEOREM.

Parallelograms conſtituted upon equal Baſes, and between the ſame Parallels, are equal between themſelves.

LET the Parallelograms ABCD, EFGH, be conſtituted upon the equal Baſes BC, FG, and between the ſame Parallels AH, BG. I ſay, the Parallelogram ABCD is equal to the Parallelogram EFGH.

For join BE, CH. Then, becauſe BC is * equal to FG, and FG to EH; BC will be likewiſe equal to EH; and they are parallel, and BE, CH, joins them. But two Right Lines joining Right Lines which are equal and parallel the ſame way, are † equal and parallel: Wherefore EBCH is a Parallelogram, and is ‡ equal to the Parallelogram ABCD; for it has the ſame Baſe BC, and is conſtituted between the ſame Parallels BC, AD. For the ſame Reaſon, the Parallelogram EFGH is equal to the ſame Parallelogram EBCH. Therefore the Parallelogram ABCD ſhall be equal to the Parallelogram EFGH. And ſo *Parallelograms conſtituted upon equal Baſes, and between the ſame Parallels, are equal between themſelves*; which was to be demonſtrated.

* *Hyp.*
† 33 *of this.*
‡ 35 *of this.*

PROPOSITION XXXVII.

THEOREM.

Triangles conſtituted upon the ſame Baſe, and between the ſame Parallels, are equal between themſelves.

LET the Triangles ABC, DBC, be conſtituted upon the ſame Baſe BC, and between the ſame Parallels AD, BC. I ſay, the Triangle ABC, is equal to the Triangle DBC.

For produce AD both ways to the Points E and F; and through B draw * BE parallel to CA; and through C, CF, parallel to BD.

* 31 *of this.*

Wherefore both EBCA, DBCF, are Parallelograms; and the Parallelogram EBCA is * equal to the Parallelogram DBCF; for they ſtand upon the ſame Baſe BC, and between the ſame Parallels BC, EF. But the Triangle ABC is † one half of the Parallelogram EBCA, becauſe the Diameter AB biſects it; and the Triangle DBC is one half of the Parallelogram DBCF; for the Diameter DC biſects it. But Things that are the Halves of equal Things, are ‡ equal between themſelves. Therefore the Triangle ABC is equal to the Triangle DBC. Wherefore, *Triangles conſtituted upon the ſame Baſe, and between the ſame Parallels, are equal between themſelves*; which was to be demonſtrated.

* 35 *of this.*
† 34 *of this.*
‡ *Ax.* 7.

PROPOSITION XXXVIII.

THEOREM.

Triangles conſtituted upon equal Baſes, and between the ſame Parallels, are equal between themſelves.

LET the Triangles ABC, DCE, be conſtituted upon the equal Baſes BC, CE, and between the ſame Parallels BE, AD. I ſay, the Triangle ABC is equal to the Triangle DCE.

For, produce AD both ways to the Points G, H; thro' B draw * BG parallel to CA; and thro' E, EH, parallel to DC.

* 31 *of this.*

Wherefore both GBCA, DCEH, are Parallelograms; and the Parallelogram GBCA is † equal to the Parallelogram DCEH: For they ſtand upon equal Baſes, BC, CE, and between the ſame Parallels BE, GH. But the Triangle ABC is is ‡ one half of the Parallelogram GBCA; for the Diameter AB biſects it; and the Triangle DCE ‡ is one half of the Parallelogram DCEH; for the Diameter DE biſects it. But Things that are the Halves of equal Things, are * equal between themſelves. Therefore the Triangle ABC is equal to the Triangle DCE. Wherefore *Triangles conſtituted upon equal Baſes, and between the ſame Parallels, are equal between themſelves*; which was to be demonſtrated.

† 36 *of this.*
‡ 34 *of this.*
* *Ax.* 7.

PRO-

PROPOSITION XXXIX.

THEOREM.

Equal Triangles conſtituted upon the ſame Baſe, on the ſame Side, are in the ſame Parallels.

LET ABC, DBC, be equal Triangles, conſtituted upon the ſame Baſe BC, on the ſame Side. I ſay, they are between the ſame Parallels. For, let AD be drawn. I ſay, AD is parallel to BC.

For, if it be not parallel, draw * the Right Line AE thro' the Point A, parallel to BC, and draw EC. * 31 *of this.*

Then the Triangle ABC † is equal to the Triangle EBC; for it is upon the ſame Baſe BC, and between the ſame Parallels BC, AE. But the Triangle ABC is ‡ equal to the Triangle DBC. Therefore the Triangle DBC is alſo equal to the Triangle EBC, a leſs to a greater, which is impoſſible. Wherefore AE is not parallel to BC: And by the ſame way of Reaſoning we prove, that no other Line but AD is parallel to BC. Therefore AD is parallel to BC. Wherefore *equal Triangles conſtituted upon the ſame Baſe, on the ſame Side, are in the ſame Parallels*; which was to be demonſtrated. † 37 *of this.* ‡ *From Hyp.*

PROPOSITION XL.

THEOREM.

Equal Triangles conſtituted upon equal Baſes, on the ſame Side, are between the ſame Parallels,

LET ABC, CDE, be equal Triangles, conſtituted upon equal Baſes BC, CE. I ſay, they are between the ſame Parallels. For, let AD be drawn. I ſay, AD is parallel to BE.

For, if it be not, let AF be drawn * thro' A, parallel to BE, and draw FE. * 31 *of this.*

Then the Triangle ABC is † equal to the Triangle FCE; for they are conſtituted upon equal Baſes, and between the ſame Parallels BE, AF. But the Triangle ABC is equal to the Triangle DCE. Therefore † 38 *of this.*

fore the Triangle D C E ſhall be equal to the Triangle F C E, the greater to the leſs, which is impoſſible. Wherefore A F is not parallel to B E. And in this manner we demonſtrate, that no Right Line can be parallel to B E, but A D. Therefore A D is parallel to B E. And ſo *equal Triangles conſtituted upon equal Baſes, on the ſame Side, are between the ſame Parallels*; which was to be demonſtrated.

PROPOSITION XLI.

THEO[redacted]REM

If a Parallelogram and a Triangle have the ſame Baſe, and are between the ſame Parallels, the Parallelogram will be double to the Triangle.

LET the Parallelogram A B C D, and the Triangle E B C, have the ſame Baſe, and be between the ſame Parallels, B C, A E. I ſay, the Paralellogram A B C D is double the Triangle E B C.

For join A C.

* 37 *of this.* Now the Triangle A B C is * equal to the Triangle E B C; for they are both conſtituted upon the ſame Baſe B C, and between the ſame Parallels B C, A E.
† 34 *of this.* But the Parallelogram A B C D is † double the Triangle A B C, ſince the Diameter A C biſects it. Wherefore likewiſe it ſhall be double to the Triangle E B C. *If*, therefore, *a Parallelogram and Triangle have both the ſame Baſe, and are between the ſame Parallels, the Parallelogram will be double to the Triangle*; which was to be demonſtrated.

PROPOSITION XLII.

PRO[redacted]BLEM.

To conſtitute a Parallelogram equal to a given Triangle, in an Angle equal to a given Right-lined Angle.

LET the given Triangle be A B C, and the Right-lin'd Angle given D. It is required to conſtitute a Parallelogram equal to the given Triangle A B C, in a Right-lin'd Angle equal to D.

Biſect * BC in E, join AE, and at the Point E, in the Right Line EC, conſtitute † an Angle CEF equal to D. Alſo draw ‡ AG thro' A, parallel to EC, and thro' C the Right Line CG parallel to FE. * 10 *of this.* † 23 *of this.* ‡ 31 *of this.*

Now FECG is a Parallelogram: And becauſe BE is equal to EC, the Triangle ABE ſhall be * equal to the Triangle AEC; for they ſtand upon equal Baſes BE, EC, and are between the ſame Parallels BC, AG. Wherefore the Triangle ABC is double to the Triangle AEC. But the Parallelogram FECG is alſo double to the Triangle AEC; for it has the ſame Baſe, and is between the ſame Parallels. Therefore the Parallelogram FECG, is equal to the Triangle ABC, and has the Angle CEF equal to the Angle D. Wherefore the Parallelogram FECG is conſtituted equal to the given Triangle ABC, in an Angle CEF equal to a given Angle D; *which was to be done.* * 38 *of this.*

PROPOSITION XLIII.

THEOREM.

In every Parallelogram the Complements of the Parallelograms that ſtand about the Diameter, are equal between themſelves.

LET ABCD be a Parallelogram, whoſe Diameter is DB; and let FH, EG, be Parallelograms ſtanding about the Diameter BD. Now AK, KC, are called the Complements of them: I ſay, the Complement AK is equal to the Complement KC.

For ſince ABCD is a Parallelogram, and BD is the Diameter thereof, the Triangle ABD * is equal to the Triangle BDC. Again, becauſe HKFD is a Parallelogram, whoſe Diameter is DK, the Triangle HDK ſhall * be equal to the Triangle DFK; and for the ſame Reaſon the Triangle KBG is equal to the Triangle KEB. But ſince the Triangle BEK is equal to the Triangle BGK, and the Triangle HDK to DFK; the Triangle BEK, together with the Triangle HDK, is equal to the Triangle BGK, together with the Triangle DFK. But the whole Triangle ABD is likewiſe equal to the whole Triangle * 34 *of this.*

BDC. Wherefore the Complement remaining, AK, will be equal to the remaining Complement KC. Therefore *in every Parallelogram the Complements of the Parallelograms, that ſtand about the Diameter, are equal between themſelves*; which was to be done.

PROPOSITION XLIV.

PROBLEM.

To apply a Parallelogram to a given Right Line, equal to a given Triangle, in a given Right-lined Angle.

LET the Right Line given be AB, the given Triangle C, and the given Right-lined Angle D. It is required, to the given Right Line AB, to apply a Parallelogram equal to the given Triangle C.

In an Angle equal to D, make the Parallelogram
* 42 *of this.* BEFG equal to * the Triangle C; in the Angle
EBG, equal to D. Place BE in a ſtrait Line with
AB, and produce FG to H, and thro' A let AH be
† 31 *of this.* drawn † parallel to either GB, or FE, and join HB.

Now, becauſe the Right Line HF falls on the Pa-
‡ 29 *of this.* rallels AH, EF, the Angles AHF, HFE, are ‡ equal
to two Right Angles. And ſo BHF, HFE, are
leſs than two Right Angles; but Right Lines making
leſs than two Right Angles, with a third Line being
* *Ax.* 12. infinitely produced, will meet * each other. Where-
fore HB, FE, produced, will meet each other; which
* 31 *of this.* let be in K, thro' which * draw KL parallel to EA, or
FH, and produce AH, GB, to the Points L and M.

Therefore HLKF is a Parallelogram, whoſe Dia-
meter is HK; and AG, ME, are Parallelograms
about HK; whereof LB, BF, are the Complements.
† 43 *of this.* Therefore LB is † equal to BF. But BF is alſo
equal to the Triangle C. Wherefore likewiſe LB
ſhall be equal to the Triangle C; and becauſe the
‡ 15 *of this.* Angle GBE is ‡ equal to the Angle ABM, and
alſo equal to the Angle D, the Angle ABM ſhall be
equal to the Angle D. Therefore to the given Right
Line AB is applied a Parallelogram, equal to the given
Triangle C, in the Angle ABM, equal to the given
Angle D; *which was to be done.*

PRO-

PROPOSITION XLV.

PROBLEM.

To make a Parallelogram equal to a given Right-lin'd Figure, in a given Right-lin'd Angle.

LET ABCD be the given Right-lined Figure, and E the Right-lined Angle given. It is required to make a Parallelogram equal to the Right-lined Figure ABCD in an Angle equal to E.

Let DB be joined, and make * the Parallelogram * 42 *of this.* FH equal to the Triangle ADB, in an Angle HKF, equal to the given Angle E.

Then to the Right Line GH apply † the Parallelogram † 44 *of this.* GM, equal to the Triangle DBC, in an Angle GHM, equal to the Angle E.

Now, becauſe the Angle E is equal to HKF, or GHM, the Angle HKF ſhall be equal to GHM, and KHG to both; and the Angles HKF, KHG, are, together, equal to the Angles KHG, GHM. But HKF, KHG, are ‡, together, equal to two Right ‡ 29 *of this.* Angles. Wherefore, likewiſe, the Angles KHG, GHM, ſhall be equal to two Right Angles: And ſo, at the given Point H in the Right Line GH, two Right Lines KH, HM, not drawn on the ſame Side, make the adjacent Angles, both together, equal to two Right Angles; and conſequently KH, HM *, * 41 *of this.* make one ſtrait Line. And becauſe the Right Line HG falls upon the Parallels KM, FG, the alternate Angles MHG, HGF, are ‡ equal. And if HGL be added to both, the Angles MHG, HGL, together, are equal to the Angles HGF, HGL, together. But the Angles MHG, HGL, are * together equal * 29 *of this.* to two Right Angles. Wherefore, likewiſe, the Angles HGF, HGL, are together equal to two Right Angles; and ſo FG, GL, make one ſtrait Line. And ſince KF is equal and parallel to HG, as likewiſe HG to ML, KF ſhall be † equal and parallel † 30 *of this.* to ML, and the Right Lines KM, FL, join them. Wherefore KM, FL, are ‡ equal and parallel. There- ‡ 34 *of this.* fore KFLM is a Parallelogram. But ſince the Triangle ABD is equal to the Parallelogram HF, and the Triangle DBC to the Parallelogram GM; then the whole Right-lined Figure ABCD will be equal to the whole Parallelogram KFLM. Therefore the Parallelogram KFLM is made equal to the given Right-lined Figure ABCD, in an Angle FKM, equal to the given Angle E; *which was to be done.*

Coroll. It is manifeſt, from what has been ſaid, how to apply a Parallelogram to a given Right Line, equal to a given Right-lined Figure in a given Right-lined Angle.

PROPOSITION XLVI.

PROBLEM.

To deſcribe a Square upon a given Right Line.

LET AB be the Right Line given, upon which it is required to deſcribe a Square.

* 11 *of this.* Draw * AC at Right Angles to AB from the Point
† 3 *of this.* A given therein; make † AD equal to AB, and thro'
‡ 31 *of this.* the Point D draw ‡ DE parallel to AB; alſo thro' B draw BE parallel to AD.

* 34 *of this.* Then ADEB is a Parallelogram; and ſo AB * is equal to DE, and AD to BE. But BA is equal to AD. Therefore the four Sides BA, AD, DE, EB, are equal to each other.

And ſo the Parallelogram ADEB is equilateral: I ſay, it is likewiſe equiangular. For becauſe the Right Line AD falls upon the Parallels AB, DE, the An-
† 29 *of this.* gles BAD, ADE, are † equal to two Right Angles. But BAD is a Right Angle: Wherefore ADE is alſo a Right Angle; but the oppoſite Sides and oppo-
‡ 34 *of this.* ſite Angles of Parallelograms are ‡ equal. Therefore each of the oppoſite Angles ABE, BED, are Right Angles; and conſequently ADBE is a Rectangle: But it has been proved to be equilateral. Therefore it is neceſſarily a Square, and is deſcribed upon the Right Line AB; *which was to be done.*

Coroll. Hence every Parallelogram, that has one Right Angle, is a Rectangle.

PRO-

PROPOSITION XLVII.

THEOREM.

In any Right-angled Triangle, the Square deſcribed upon the Side ſubtending the Right Angle, is equal to both the Squares deſcribed upon the Sides containing the Right Angle.

LET ABC be a Right-angled Triangle, having the Right Angle BAC. I ſay, the Square deſcribed upon the Right Line BC, is equal to both the Squares deſcribed upon the Sides BA, AC.

For deſcribe * upon BC the Square BDEC, and on BA, AC, the Squares GB, HC, and thro' the Point A draw AL parallel to BD, or CE; and let AD, FC, be joined. * 46 *of this.*

Then, becauſe the Angles BAC, BAG, † are Right ones, two Right Lines AG, AC, at the given Point A, in the Right Line BA, being on contrary Sides thereof, make the adjacent Angles equal to two Right Angles. Therefore CA, AG, make ‡ one ſtrait Line; by the ſame Reaſon AB, AH, make one ſtrait Line. And ſince the Angle DBC is equal to the Angle FBA, for each of them is a Right one, add ABC, which is common, and the whole Angle DBA is * equal to the whole Angle FBC. And ſince the two Sides AB, BD, are equal to the two Sides FB, BC, each to each, and the Angle DBA equal to the Angle FBC; the Baſe AD will be † equal to the Baſe FC, and the Triangle ABD equal to the Triangle FBC: But the Parallelogram BL is ‡ double to the Triangle ABD; for they have the ſame Baſe DB, and are between the ſame Parallels BD, AL. The Square GB is ‡ alſo double to the Triangle FBC; for they have the ſame Baſe FB, and are in the ſame Parallels FB, GC. But Things that are the Doubles of equal Things, are * equal to each other. Therefore the Parallelogram BL is equal to the Square GB. After the ſame manner, AE, BK, being joined, we prove, that the Parallelogram CL is equal to the Square HC. Therefore the whole Square DBEC is equal to the two Squares GB, HC. But the Square DBEC is deſcribed on the Right Line BC,

† *Def.* 30.
‡ 14 *of this.*
* *Ax.* 2.
† 4 *of this.*
‡ 41 *of this.*
* *Ax.* 6.

B C, and the Squares G B, H C, on B A, A C. Therefore the Square B E, deſcribed on the Side B C, is equal to the Squares deſcribed on the Sides B A, A C. Wherefore *in any Right-angled Triangle, the Square deſcribed upon the Side ſubtending the Right Angle, is equal to both the Squares deſcribed upon the Sides containing the Right Angle.*

PROPOSITION XLVIII.

THEOREM.

If a Square deſcribed upon one Side of a Triangle be equal to the Squares deſcribed upon the other two Sides of the ſaid Triangle, then the Angle contained by theſe two other Sides is a Right Angle.

IF the Square deſcribed upon the Side B C of the Triangle A B C, be equal to the Squares deſcribed upon the other two Sides of the Triangle B A, A C: I ſay, the Angle B A C is a Right one.

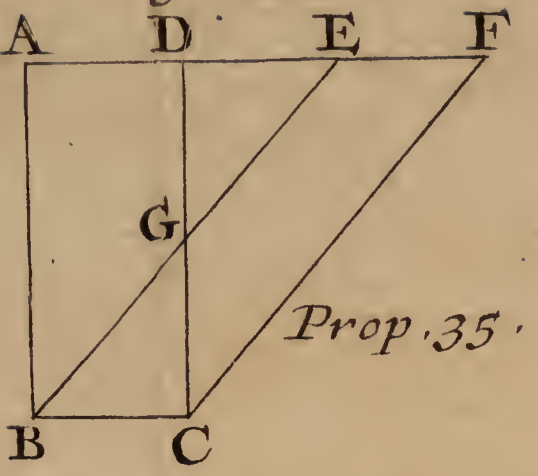
For, let there be drawn A D from the Point A, at Right Angles, to A C; likewiſe make A D equal to B A, and join D C.

Then, becauſe D A is equal to A B, the Square deſcribed on D A will be equal to the Square deſcribed on A B. And adding the common Square deſcribed on A C, the Squares deſcribed on D A, A C, are equal to the Squares deſcribed on B A, A C. But the Square deſcribed on D C is * equal to the Squares deſcribed on D A, A C; for D A C is a Right Angle: But the Square on B C is put equal to the Squares on B A, A C. Therefore the Square deſcribed on D C is equal to the Square deſcribed on B C; and ſo the Side C D is equal to the Side C B. And becauſe D A is equal to A B, and A C is common, the two Sides D A, A C, are equal to the two Sides B A, A C; and the Baſe D C is equal to the Baſe C B. Therefore the Angle D A C is ‡ equal to the Angle B A C; but D A C is a Right Angle; and ſo B A C will be a Right Angle alſo. *If*, therefore, *a Square deſcribed upon one Side of a Triangle be equal to the Squares deſcribed upon the other two Sides of the ſaid Triangle, then the Angle contained by theſe two other Sides is a Right Angle*; which was to be demonſtrated.

* 47 *of this.*

‡ 8 *of this.*

Book 1 Plate 3.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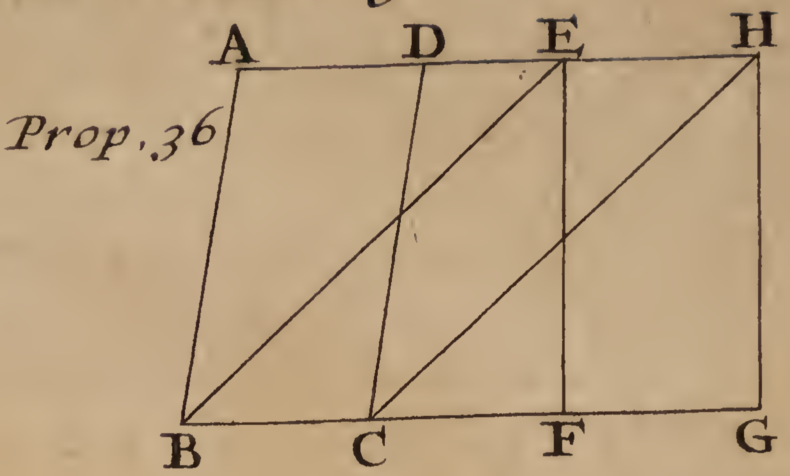
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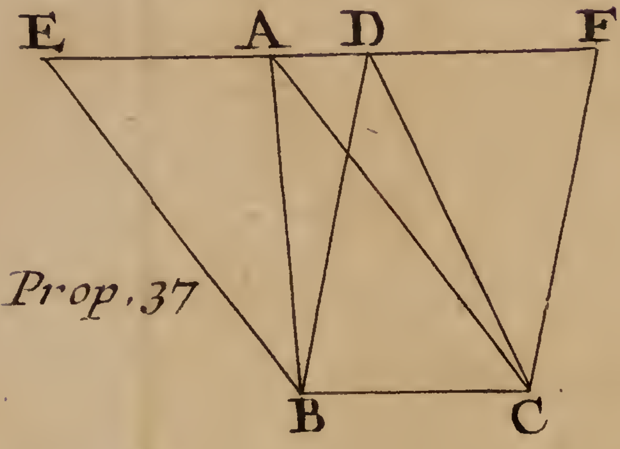
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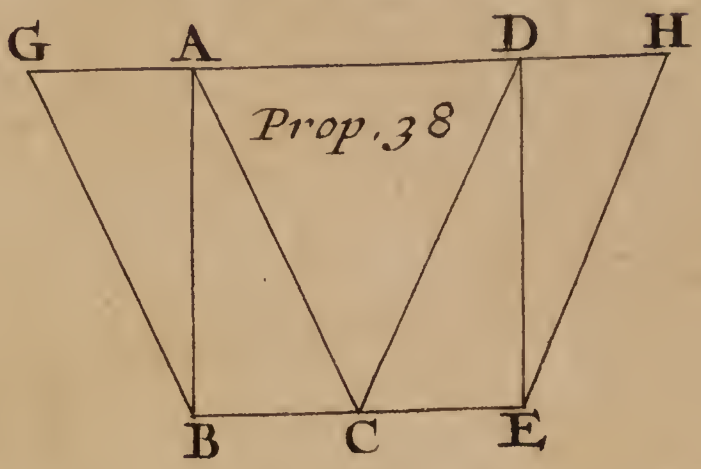
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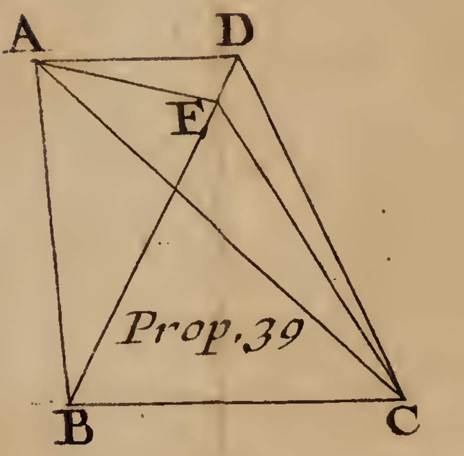
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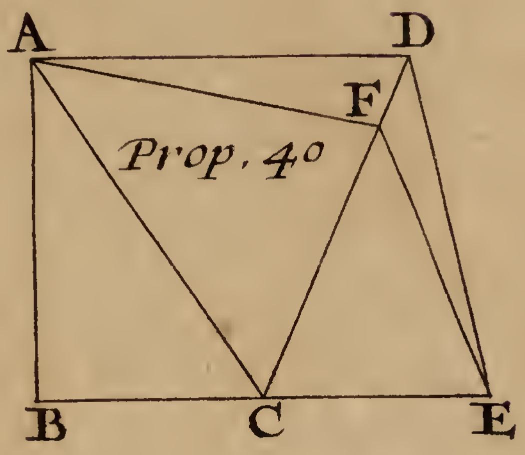
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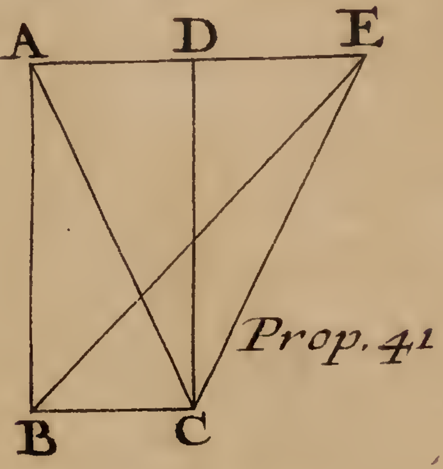
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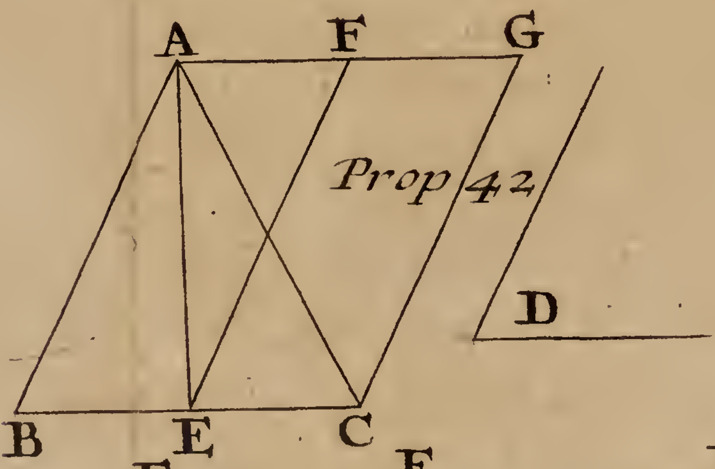
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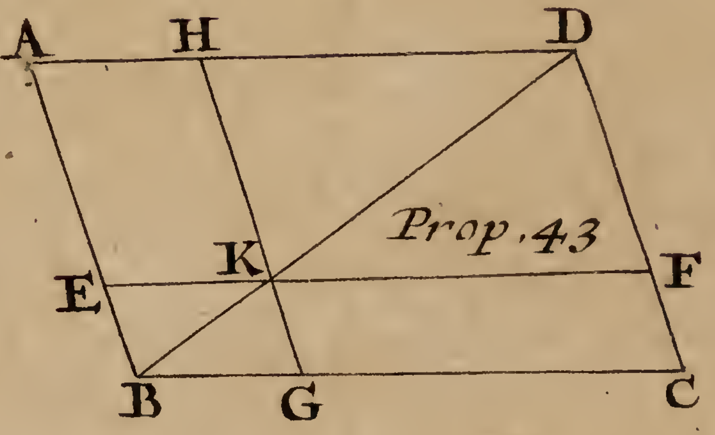
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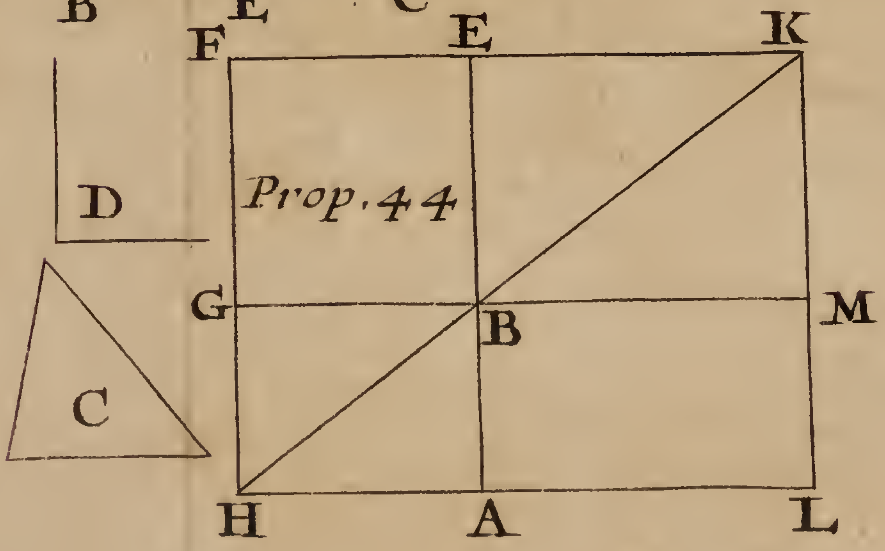
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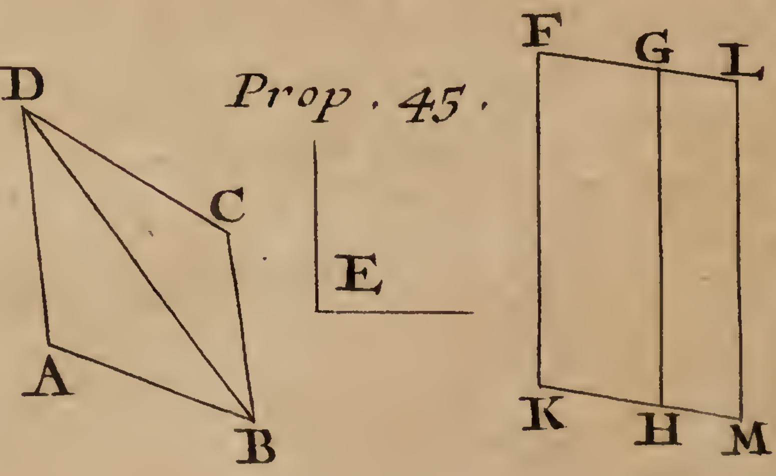
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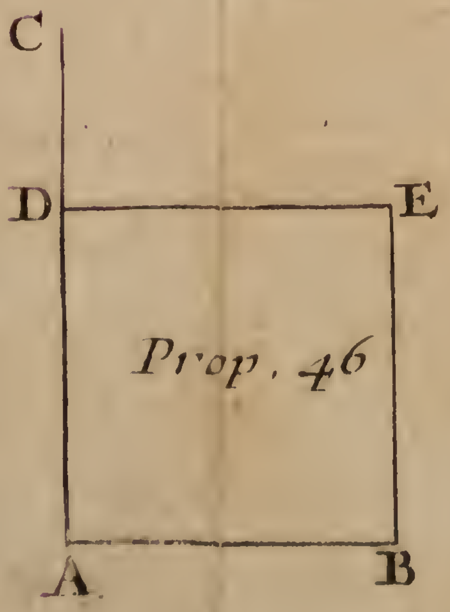
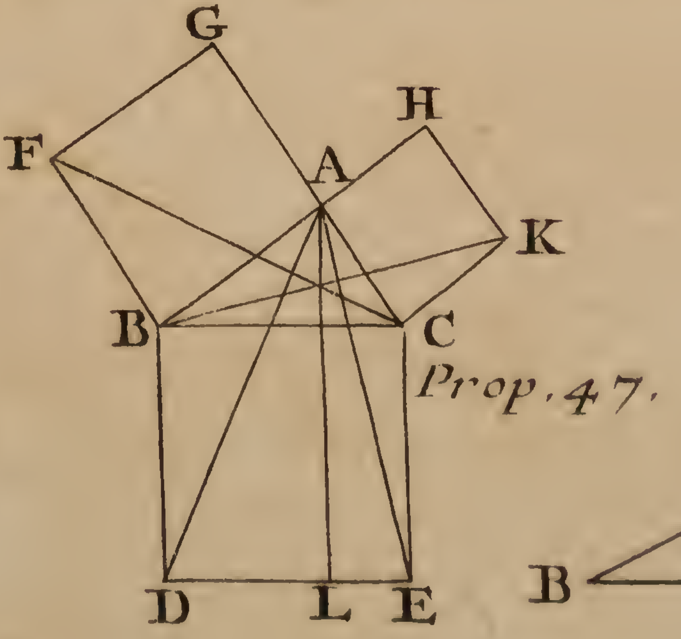
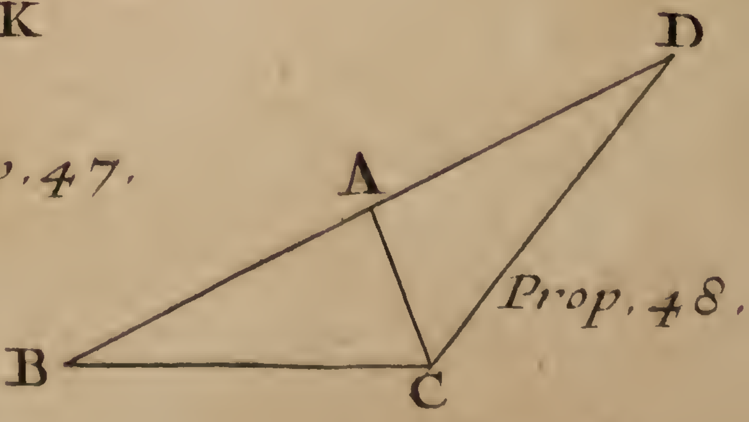

Prop. 46


Prop. 47.


Prop. 48.

EUCLID's ELEMENTS.

BOOK II.

DEFINITIONS.

I. *EVERY Right-angled Parallelogram is ſaid to be contained under two Right Lines, comprehending a Right Angle.*

II. *In every Parallelogram, either of thoſe Parallelograms that are about the Diameter, together with the Complements, is called a Gnomon.*

PROPOSITION I.

THEOREM.

If there be two Right Lines, and one of them be divided into any Number of Parts; the Rectangle comprehended under the whole, and divided Line, ſhall be equal to all the Rectangles contained under the whole Line, and the ſeveral Segments of the divided Line.

LET A and BC be two Right Lines, whereof BC is cut or divided any how in the Points D, E. I ſay, the Rectangle contained under the Right Lines A and BC, is equal to the Rectangles contained under A and BD, A and DE, and A and EC.

* 11. 1. † 3. 1. ‡ 31. 1. For, let * BF be drawn from the Point B, at Right Angles, to BC; and make † BG equal to A; and let ‡ GH be drawn thro' G parallel to BC: Likewiſe let ‡ there be drawn DK, EL, CH, thro' D, E, C, parallel to BG.

Then the Rectangle BH is equal to the Rectangles BK, DL, EH; but the Rectangle BH is that contained under A and BC; for it is contained under GB, BC; and GB is equal to A; and the Rectangle BK is that contained under A and BD; for it is contained under GD and BD, and GB is equal to A; and the Rectangle DL is that contained under A and DE, becauſe DK, that is, BG, is equal to A: So likewiſe the Rectangle EH is that contained under A and EC. Therefore the Rectangle under A and BC, is equal to the Rectangle under A and BD, A and DE, and A and EC. Therefore, *if there be two Right Lines given, and one of them be divided into any Number of Parts, the Rectangle comprehended under the whole, and divided Line, ſhall be equal to all the Rectangles contained under the whole Line, and the ſeveral Segments of the divided Line*; which was to be demonſtrated.

PROPOSITION II.

THEOREM.

If a Right Line be any how divided, the Rectangles contained under the whole Line, and each of the Segments, or Parts, are equal to the Square of the whole Line.

LET the Right Line AB be any how divided in the Point C. I ſay, the Rectangle contained under AB, BC, together with that contained under AB and AC, is equal to the Square made on AB.

For let the Square ADEB be deſcribed * on AB, *46. 1. and thro' C let CF be drawn parallel to AD or BE. Therefore AE is equal to the Rectangles AF and CE. But AE is a Square deſcribed upon AB; and AF is the Rectangle contained under BA, AC; for it is contained under DA and AC, whereof AD is equal to AB; and the Rectangle CE is contained under AB, BC, ſince BE is equal to AB. Wherefore the Rectangle under AB and AC, together with the Rectangle under AB and BC, is equal to the Square of AB. Therefore, *if a Right Line be any how divided, the Rectangles contained under the whole Line, and each of the Segments, or Parts, are equal to the Square of the whole Line.*

PROPOSITION III.

THEOREM.

If a Right Line be any how cut, the Rectangle contained under the whole Line, and one of its Parts, is equal to the Rectangle contained under the two Parts together, with the Square of the firſt-mentioned Part.

LET the Right Line AB be any how cut in the Point C. I ſay, the Rectangle under AB and BC is equal to the Rectangle under AC and BC, together with the Square deſcribed on BC.

For

* 46. 1. For deſcribe * the Square CDEB upon BC; † 31. 1. produce ED to F; and let AF be drawn † thro' A, parallel to CD or BE.

Then the Rectangle AE ſhall be equal to the two Rectangles AD, CE: And the Rectangle AE is that contained under AB and BC; for it is contained under AB and BE, whereof BE is equal to BC: And the Rectangle AD is that contained under AC and CB, ſince DC is equal to CB: And DB is a Square deſcribed upon BC. Wherefore the Rectangle under AB and BC is equal to the Rectangle under AC and CB, together with the Square deſcribed upon BC. Therefore, *if a Right Line be any how cut, the Rectangle contained under the whole Line, and one of its Parts, is equal to the Rectangle contained under the two Parts together, with the Square of the firſt-mentioned Parts*; which was to be demonſtrated.

PROPOSITION IV.

THEOREM.

If a Right Line be any how cut, the Square which is made on the whole Line, will be equal to the Squares made on the Segments thereof, together with twice the Rectangle contained under the Segments.

LET the Right Line AB be any how cut in C. I ſay, the Square made on AB is equal to the Squares of AC, CB, together, with twice the Rectangle contained under AC, CB.

* 46. 1. For*deſcribe the Square ADEB upon AB, join BD, † 31. 1. and thro' C draw † CGF parallel to AD or BE; and alſo thro' G draw HK parallel to AB or DE.

Then, becauſe CF is parallel to AD, and BD falls ‡ 29. 1. upon them, the outward Angle BGC ſhall be ‡ equal to the inward and oppoſite Angle ADB; but the * 5. 1. Angle ADB is * equal to the Angle ABD, ſince the Side BA is equal to the Side AD. Wherefore the Angle CGB is equal to the Angle GBC; and ſo the † 6. 1. Side BC equal † to the Side CG; but likewiſe the ‡ 34. 1. Side CB is ‡ equal to the Side GK, and the Side CG to BK. Therefore GK is equal to KB, and CGKB is equilateral. I ſay, it is alſo Right-angled; for, becauſe CG is parallel to BK, and CB falls on them, the Angles KBC, GCB, are ‡ equal to two Right Angles. But KBC is a Right Angle. Wherefore GBC alſo is a Right Angle, and the oppoſite Angles GCB, CGK, GKB, ſhall be Right Angles. Therefore CGKB is a Rectangle. But it has been proved to be equilateral. Therefore CGKB is a Square deſcribed upon BC. For the ſame Reaſon HF is alſo a Square made upon HG, that is equal to the Square of AC. Wherefore HF and CK are the Squares of AC and CB. And becauſe the Rectangle AG is * equal to the Rectangle GE, and AG is that which * 43. 1. is contained under AC and CB; for GC is equal to CB: GE ſhall be equal to the Rectangle under AC, and CB. Wherefore the Rectangles AG, GE, are equal to twice the Rectangle contained under AC, CB; and HF, CK, are the Squares of AC, CB. Therefore the four Figures HF, CK, AG, GE, are equal to the Squares of AC and CB, with twice the Rectangle contained under AC and CB. But HF, CK, AG, GE, make up the whole Square of AB, *viz.* ADEB. Therefore the Square of AB is equal to the Squares of AC, CB, together with twice the Rectangle contained under AC, CB. Wherefore, *if a Right Line be any how cut, the Square which is made on the whole Line, will be equal to the Squares made on the Segments thereof, together with twice the Rectangle contained under the Segments*; which was to be demonſtrated.

Coroll. Hence it is manifeſt, that the Parallelograms which ſtand about the Diameter of a Square, are likewiſe Squares.

PRO-

PROPOSITION V.

THEOREM.

If a Right Line be cut into two equal Parts, and into two unequal ones; the Rectangle under the unequal Parts, together with the Square that is made of the intermediate Diſtance, is equal to the Square made of half the Line.

LET any Right Line AB be cut into two equal Parts in C, and into two unequal Parts in D. I ſay, the Rectangle contained under AD, DB, together with the Square of CD, is equal to the Square of BC.

† 46. 1. For † deſcribe CEFB, the Square of BC; draw BE, and thro' D draw * DHG, parallel to CE, or BF, and thro' H draw KLO, parallel to CB, or EF, and AK thro' A, parallel to CL, or BO. * 31. 1.

‡ 43. 1. Now the Complement CH is ‡ equal to the Complement HF. Add DO, which is common to both of them, and the whole CO, is equal to the whole DF: But CO is equal to AL, becauſe AC is equal to CB; therefore AL is equal to DF, and adding CH, which is common, the whole AH ſhall be equal to FD, DL, together. But AH is the Rectangle contained under AD, DB; for DH is * equal to DB, and FD, DL, is the Gnomon MNX; therefore MNX is equal to the Rectangle contained under AD, DB; and if LG, being common, and * equal to the Square of CD, be added, then the Gnomon MNX, and LG, are equal to the Rectangle contained under AD, DB, together with the Square of CD; but the Gnomon MNX, and LG, make up the whole Square CEFB, *viz.* the Square of CB. Therefore the Rectangle under AD, DB, together with the Square of CD, is equal to the Square of CB. Wherefore, *if a Right Line be cut into two equal Parts, and into two unequal ones; the Rectangle under the unequal Parts, together with the Square that is made of the intermediate Diſtance, is equal to the Square made of half the Line*; which was to be demonſtrated. * *Cor.* 4. *of this.*

PRO-

PROPOSITION VI.

THEOREM.

If a Right Line be divided into two equal Parts, and another Right Line be added directly to the ſame, the Rectangle contained under the Line, compounded of the whole and added Line, (taken as one Line) and the added Line, together with the Square of half the Line, is equal to the Square of the Line compounded of half the Line, and the added Line taken as one Line.

LET the Right Line AB be biſected in the Point C, and BD added directly thereto. I ſay, the Rectangle under AD, and DB, together with the Square of BC, is equal to the Square of CD.

For, deſcribe * CEFD, the Square of CD, and * 46. 1.
join DE; draw † BHG thro' B, parallel to CE, or † 31. 1.
DF, and KLM thro' H, parallel to AD, or EF, as
alſo AK thro' A, parallel to CL, or DM.

Then, becauſe AC is equal to CB, the Rectangle
AL ſhall be * equal to the Rectangle CH; but CH * 36. 1.
is ‡ equal to HF. Therefore AL will be equal to ‡ 43. 1.
HF; and adding CM, which is common to both,
then the whole Rectangle AM is equal to the
Gnomon NXO. But AM is that Rectangle which
is contained under AD, DB; for DM is * equal to * *Cor.* 4.
DB; therefore the Gnomon NXO is equal to *of this.*
the Rectangle under AD, and DB. And adding
LG, which is common, *viz.* † the Square of CB; † *Cor.* 4.
and then the Rectangle under AD, DB, together *of this.*
with the Square of BC, is equal to the Gnomon
NXO with LG. But the Gnomon NXO, and
LG, together, make up the Figure CEFD, that
is, the Square of CD. Therefore the Rectangle under AD, and DB, together with the Square of
BC, is equal to the Square of CD. Therefore, *if
a Right Line be divided into two equal Parts, and
another Right Line be added directly to the ſame, the
Rectangle contained under the Line, compounded of the
whole and added Line, (taken as one Line) and the*

added Line, together with the Square of half the Line, is equal to the Square of the Line compounded of half the Line, and the added Line taken as one Line; which was to be demonſtrated.

PROPOSITION VII.

THEOREM.

If a Right Line be any how cut, the Square of the whole Line, together with the Square of one of the Segments, is equal to double the Rectangle contained under the whole Line, and the ſaid Segment, together with the Square, made of the other Segment.

LET the Right Line AB be any how cut in the Point C. I ſay, the Squares of AB, BC, together, are equal to double the Rectangle contained under AB, BC, together with the Square, made of AC.

* 46. 1. For let the Square of AB be * deſcribed, *viz.* ADEB, and conſtruct § the Figure.

† 43. 1. Then, becauſe the Rectangle AG is † equal to the Rectangle GE; if CF, which is common, be added to both, the whole Rectangle AF ſhall be equal to the whole Rectangle CE; and ſo the Rectangles AF, CE, are double to the Rectangle AF; but AF, CE, make up the Gnomon KLM, and the Square CF. Therefore the Gnomon KLM, together with the Square CF, ſhall be double to the Rectangle AF. But double the Rectangle under AB,
‡ *Cor.* 4. BC, is double the Rectangle AF; for BF is ‡ equal to BC. Therefore the Gnomon KLM, and the Square CF, are equal to twice the Rectangle contained under AB, BC. And if HF, which is common, being the Square of AC, be added to both; then the Gnomon KLM, and the Squares CF, HF, are equal to double the Rectangle contained under

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§ *A Figure is ſaid to conſtructed, when Lines, drawn in a Parallelogram parallel to the Sides thereof, cut the Diameter in one Point, and make two Parallelograms about the Diameter, and two Complements. So likewiſe a double Figure is ſaid to be conſtructed, when two Right Lines parallel to the Sides, make four Parallelograms about the Diameter, and four Complements.*

under AB, BC, together with the Square of AC. But the Gnomon KLM, together with the Squares CF, HF, are equal to ADEB, and CF, *viz.* the Squares of AB, BC. Therefore the Squares of AB, BC, are together equal to double the Rectangle contained under AB, BC, together with the Square of AC. Therefore, *if a Right Line be any how cut, the Square of the whole Line, together with the Square of one of the Segments, is equal to double the Rectangle contained under the whole Line, and the ſaid Segment, together with the Square, made of the other Segment;* which was to be demonſtrated.

PROPOSITION VIII.

THEOREM.

If a Right Line be any how cut into two Parts, four times the Rectangle contained under the whole Line, and one of the Parts, together with the Square of the other Part, is equal to the Square of the Line, compounded of the whole Line, and the firſt Part taken as one Line.

LET the Right Line AB be cut any how in C. I ſay, four times the Rectangle contained under AB, BC, together with the Square of AC, is equal to the Square of AB, and BC, taken as one Line.

For, let the Right Line AB be produced to D, ſo that BD be equal to BC; deſcribe the Square AEFD, on AD, and conſtruct the double Figure.

Now, ſince CB is * equal to BD, and alſo to † GK, and BD is equal to KN; GK ſhall be likewiſe equal to KN; by the ſame Reaſoning, PR is equal to RO. And ſince CB is equal to BD, and GK to KN, the Rectangle CK will ‡ be equal to the Rectangle BN, and the Rectangle GR to the Rectangle RN. But CK is * equal to RN; for they are the Complements of the Parallelogram CO. Therefore BN is equal to GR, and the four Squares BN, KC, GR, RN, are equal to each other; and ſo they are together Quadruple CK. Again, becauſe CB is equal to BD, and BD to BK, that is, equal to CG; and the ſaid CB is equal alſo

* *Hyp.*
† 34. 1.
‡ 36. 1.
* 43. 1.

alſo to G K, that is, to G P; therefore C G ſhall be equal to G P. But P R is equal to R O; therefore the Rectangle A G ſhall be equal to the Rectangle M P, and the Rectangle P L equal to R F. But M P is equal to P L; for they are the Complements of the Parallelogram M L. Wherefore A G is equal alſo to R F. Therefore the four Parallelograms A G, M P, P L, R F, are equal to each other, and accordingly they are together Quadruple of A G. But it has been proved, that the four Squares C K, B N, G R, R N, are Quadruple of C K. Therefore the four Rectangles, and the four Squares, makeing up the Gnomon S T Y, are together Quadruple of A K; and becauſe A K is a Rectangle contained under A B, and B C, for B K is equal to B C; four times the Rectangle under A B, B C, will be Quadruple of A K. But the Gnomon S T Y has been proved to be Quadruple of A K. And ſo four times the Rectangle contained under A B, B C, is equal to the Gnomon S T Y. And if X H, being equal to † the Square of A C, which is common, be added to both; then four times the Rectangle contained under A B, B C, together with the Square of A C, is equal to the Gnomon S T Y, and the Square X H. But the Gnomon S T Y and H X, make A E F D, the whole Square of A D. Therefore four times the Rectangle contained under A B, B C, together with the Square of A C, is equal to the Square of A D, that is, of A B and B C taken as one Line. Wherefore, *if a Right Line be any how cut into two Parts, four times the Rectangle contained under the whole Line, and one of the Parts, together with the Square of the other Part, is equal to the Square of the Line, compounded of the whole Line, and the firſt Part taken as one Line*; which was to be demonſtrated.

† *Cor. 4. of this.*

PRO-

PROPOSITION IX.

THEOREM.

If a Right Line be any how cut into two equal, and two unequal Parts; then the Squares of the unequal Parts together, are double to the Square of the half Line, and the Square of the intermediate Part.

LET any Right Line AB be cut unequally in D, and equally in C. I ſay, the Squares of AD, DB, together, are double to the Squares of AC and CD together.

For, let * CE be drawn from the Point C at Right * 11. 1.
Angles to AB, which make equal to AC, or CB;
and join EA, EB. Alſo thro' D let † DF be drawn † 31. 1.
parallel to CE, and FG thro' F parallel to AB, and
draw AF.

Now, becauſe AC is equal to CE, the Angle EAC
will be ‡ equal to the Angle AEC; and ſince the ‡ 5. 1.
Angle at C is a Right one, the other Angles, AEC,
EAC, together, ſhall * make one Right Angle, and * 3. Cor.
are equal to each other: And ſo AEC, EAC, are each 32. 1.
equal to half a Right Angle. For the ſame Reaſon
are alſo CEB, EBC, each of them half Right Angles.
Therefore the whole Angle AEB is a Right Angle.
And ſince the Angle GEF is half a Right one,
and EGF is a Right Angle; for it is † equal to the † 29. 1.
inward and oppoſite Angle ECB; the other Angle
EFG will be alſo equal to half a Right one. Therefore the Angle GEF is equal to the Angle EFG.
And ſo the Side EG is ‡ equal to the Side GF. Again, ‡ 6. 1.
becauſe the Angle at B is half a Right one, and FDB
is a Right one, becauſe equal to the inward and oppoſite Angle ECB, the other Angle BFD will be half a Right Angle. Therefore the Angle at B is equal to the Angle BFD; and ſo the Side DF is equal to the Side DB. And becauſe AC is equal to CE, the Square of AC will be equal to the Square of CE. Therefore the Squares of AC, CE, together, are double to the Square of AC; but the Square of EA
is † equal to the Squares of AC, CE, together, ſince † 47. 1.

ACE is a Right Angle. Therefore the Square of EA is double to the Square of AC. Again, becauſe EG is equal to GF, and the Square of EG is equal to the Square of GF; therefore the Squares of EG, GF, together, are double to the Square of GF. But the Square of EF is † equal to the Squares of EG, GF. † 47. 1. Therefore the Square of EF is double the Square of GF: But GF is equal to CD; and ſo the Square of EF double to the Square of CD. But the Square of AE is likewiſe double to the Square of AC. Wherefore the Squares of AE, and EF, are double to the Squares of AC and CD. But the Square of AF is † equal to the Squares of AE and EF; becauſe the Angle AEF is a Right Angle, and conſequently the Square of AF is double to the Squares of AC, and CD. But the Squares of AD, DF, are equal to the Square of AF: For the Angle at D is a Right Angle. Therefore the Squares of AD, and DF, together, ſhall be double to the Squares of AC and CD together. But DF is equal to DB. Therefore the Squares of AD, and DB, together, will be double to the Squares of AC and CD, together. Wherefore, *if a Right Line be any how cut into two equal, and two unequal Parts; then the Squares of the unequal Parts together, are double to the Square of the half Line, and the Square of the intermediate Part*; which was to be demonſtrated.

PROPOSITION X.

THEOREM.

If a Right Line be cut into two equal Parts, and to it be directly added another; the Square made on [the Line compounded of] the whole Line, and the added one, together with the Square of the added Line, ſhall be double to the Square of the half Line, and the Square of [that Line which is compounded of] the half, and the added Line.

LET the Right Line AB be biſected in C, and any ſtrait Line BD added directly thereto. I ſay, the Squares of AD, DB, together, are double to the Squares of AC, CD, together.

For, draw * C E from the Point C at Right Angles to A B, which make equal to A C, or C B, and draw A E, E B; likewiſe thro' E let E F be † drawn parallel to A D, and thro' D, D F † parallel to C E. * 11. 1. † 31. 1.

Then, becauſe the Right Line E F falls upon the Parallels E C, F D, the Angles C E F, E F D, are ‡ equal to two Right Angles. Therefore the Angles F E B, E F D, are together leſs than two Right Angles. But Right Lines making, with a third Line, Angles together leſs than two Right Angles, being infinitely produced, will meet *. Wherefore E B, F D, produced, will meet towards B D. Now, let them be produced, and meet each other in the Point G, and let A G be drawn. ‡ 29. 1. * *Ax.* 12.

And then, becauſe A C is equal to C E, the Angle A E C will be equal to the Angle E A C †: But the Angle at C is a Right Angle. Therefore the Angle C A E, or A E C, is half a Right one. By the ſame way of Reaſoning, the Angle C E B, or E B C, is half a Right one. Therefore A E B is a Right Angle. And ſince E B C is half a Right Angle, D B G will ‡ alſo be half a Right Angle, ſince it is vertical to C B E. But B D G is a Right Angle alſo; for it is * equal to the alternate Angle D C E. Therefore the remaining Angle D G B is half a Right Angle, and ſo equal to D B G. Wherefore the Side B D is † equal to the Side D G. Again, becauſe E G F is half a Right Angle, and the Angle at F is a Right Angle, for it is equal to the oppoſite Angle at C; the remaining Angle F E G will be alſo half a Right one, and is equal to the Angle E G F; and ſo the Side G F is † equal to the Side E F. And ſince E C is equal to C A, and the Square of E C equal to the Square of C A; therefore the Squares of E C, C A, together, are double to the Square of C A. But the Square of E A is ‡ equal to the Squares of E C, C A. Wherefore the Square of E A is double to the Square of A C. Again, becauſe G F is equal to F E, the Square of G F alſo is equal to the Square of F E. Wherefore the Squares of G F, F E, are double to the Square of F E. But the Square of E G is † equal to the Squares of G F, F E. Therefore the Square of E G is double to the Square of E F: But E F is equal to C D. Wherefore the Square of E G ſhall be double to the Square of C D. † 5. 1. ‡ 15. 1. * 29. 1. † 6. 1. ‡ 47. 1.

But the Square of EA has been proved to be double to the Square of AC. Therefore the Squares of AE, EG, are double the Squares of AC, CD. But the
† 47. 1. Square of AG is † equal to the Squares of AE, EG; and conſequently the Square of AG is double to the Squares of AC, CD. But the Squares of AD, DG, are † equal to the Square AG. Therefore the Squares of AD, DG, are double the Squares of AC, CD. But DG is equal to DB. Wherefore the Squares of AD, DB, are double to the Squares of AC, CD. Therefore, *if a Right Line be cut into two equal Parts, and to it be directly added another; the Square made on [the Line compounded of] the whole Line, and the added one, together with the Square of the added Line, ſhall be double to the Square of the half Line, and the Square of [that Line which is compounded of] the half, and the added Line.*

PROPOSITION XI.

PROBLEM.

To cut a given Right Line ſo, that the Rectangle contained under the whole Line, and one Segment, be equal to the Square of the other Segment.

LET AB be a given Right Line. It is required to cut the ſame ſo, that the Rectangle contained under the whole, and one Segment thereof, be equal to the Square of the other Segment.

* 46. 1. Deſcribe * ABCD the Square of AB; biſect AC in E, and draw BE: Alſo produce CA to F, ſo that EF be equal to EB. Deſcribe FGHA the Square of AF, and produce GH to K. I ſay, AB is cut in H ſo, that the Rectangle under AB, BH, is equal to the Square of AH.

For, ſince the Right Line AC is biſected in E, and AF is directly added thereto, the Rectangle under CF, FA, together with the Square of AE, will be
† 6 *of this.* † equal to the Square of EF. But EF is equal to EB. Therefore the Rectangle under CF, FA, together with the Square of AE, is equal to the Square
‡ 47. 1. of EB. But the Squares of BA, AE, are ‡ equal to the Square of EB; for the Angle at A is a Right

Angle. Therefore the Rectangle under CF, FA, together with the Square of AE, is equal to the Squares of BA, AE. And taking away the Square of AE, which is common, the remaining Rectangle under CF, FA, is equal to the Square of AB. But FK is the Rectangle under CF, FA; ſince AF is equal to FG; and the Square of AB is AD. Wherefore the Rectangle FK is equal to the Square AD. And if AK, which is common, be taken from both, then the remaining Square FH is equal to the remaining Rectangle HD. But HD is the Rectangle under AB, BH, ſince AB is equal to BD, and FH is the Square of AH. Therefore the Rectangle under AB, BH, ſhall be equal to the Square of AH. And ſo the given Right Line AB is cut in H, ſo that the Rectangle under AB, BH, is equal to the Square of AH; *which was to be done.*

PROPOSITION XII.

THEOREM.

In obtuſe-angled Triangles, the Square of the Side ſubtending the obtuſe Angle, is greater than the Squares of the Sides containing the obtuſe Angle, by twice the Rectangle under one of the Sides, containing the obtuſe Angle, viz. *that on which, produced, the Perpendicular falls, and the Line taken without, between the perpendicular and the obtuſe Angle.*

LET ABC be an obtuſe-angled Triangle, having the obtuſe Angle BAC; and * from the Point B draw BD perpendicular to the Side CA produced. I ſay, the Square of BC is greater than the Squares of BA and AC, by twice the Rectangle contained under CA, and AD.

* 12. 1.

For, becauſe the Right Line CD is any how cut in the Point A, the Square of CD ſhall be † equal to the Squares of CA, and AD, together with twice the Rectangle under CA, and AD. And if the Square of BD, which is common, be added, then the Squares of CD, DB, are equal to the Squares of CA, AD,

† 4 *of this.*

and DB, and twice the Rectangle contained under
* 47. 1. CA and AD. But the Square of CB is * equal to the Squares of CD, DB; for the Angle at D is a Right one, ſince BD is perpendicular, and the Square of AB is * equal to the Squares of AD, and DB: Therefore the Square of CB is equal to the Squares of CA and AB, together with twice a Rectangle under CA and AD. Therefore *in obtuſe-angled Triangles, the Square of the Side ſubtending the obtuſe Angle, is greater than the Squares of the Sides containing the obtuſe Angle, by twice the Rectangle under one of the Sides containing the obtuſe Angle,* viz. *that on which, produced, the Perpendicular falls, and the Line taken without, between the perpendicular and the obtuſe Angle*; which was to be demonſtrated.

PROPOSITION XIII.

THEOREM.

In acute-angled Triangles, the Square of the Side ſubtending the acute Angles, is leſs than the Squares of the Sides containing the acute Angle, by twice a Rectangle under one of the Sides about the acute Angle; viz. *on which the Perpendicular falls, and the Line aſſumed within the Triangle, from the perpendicular to the acute Angle.*

LET ABC be an acute-angled Triangle, having
* 12. 1. the acute Angle B: And from A let there * be drawn AD perpendicular to BC. I ſay, the Square of AC is leſs than the Squares of CB and BA, by twice a Rectangle under CB and BD.

For, becauſe the Right Line CB is cut any how in
† 7 *of this.* D, the Squares of CB and BD will be † equal to twice a Rectangle under CB and BD, together with the Square of DC. And if the Square of AD be added to both, then the Squares of CB, BD, and DA, are equal to twice the Rectangle contained under CB and BD, together with the Squares of AD
‡ 47. 1. and DC. But the Square of AB is ‡ equal to the Squares of BD and DA; for the Angle at D is a Right Angle. And the Square of AC is ‡ equal to the

the Squares of A D and D C. Therefore the Squares of C B and B A are equal to the Square of A C, together, with twice the Rectangle contained under C B and B D. Wherefore the Square of A C only, is leſs than the Squares of C B and B A, by twice the Rectangle under C B and B D. Therefore *in acute-angled Triangles, the Square of the Side ſubtending the acute Angles, is leſs than the Squares of the Sides containing the acute Angle, by twice a Rectangle under one of the Sides about the acute Angle*, viz. *on which the Perpendicular falls, and the Line aſſumed within the Triangle, from the perpendicular to the acute Angle*; which was to be demonſtrated.

PROPOSITION XIV.

PROBLEM.

To make a Square equal to a given Right-lined Figure.

LET A be the given Right-lined Figure. It is required to make a Square equal thereto.

Make * the Right-angled Parallelogram B C D E equal to the Right-lined Figure A. Now if B E be equal to E D, what was propoſed will be already done, ſince the Square B D is made equal to the Right-lined Figure A: But if it be not, let either B E or E D be the greater: Suppoſe B E, which let be produced to F; ſo that E F be equal to E D. This being done, let B E be † biſected in G, about which, as a Centre, with the Diſtance G B or G F, deſcribe the Semicircle B H F; and let D E be produced to H, and draw G H. Now, becauſe the Right Line B F is divided into two equal Parts in G, and into two unequal ones in E, the Rectangle under B E and E F, together with the Square of G E, ſhall be ‡ equal to the Square of G F. But G F is equal to G H. Therefore the Rectangle under B E, E F, together, with the Square of G E, is equal to the Square of G H. But the Squares of H E and E G are * equal to the Square of G H. Wherefore the Rectangle under B E, E F, together with the Square of E G, is equal to the Squares of H E, E G. And if the Square of E G, which is common,

* 45. 1.

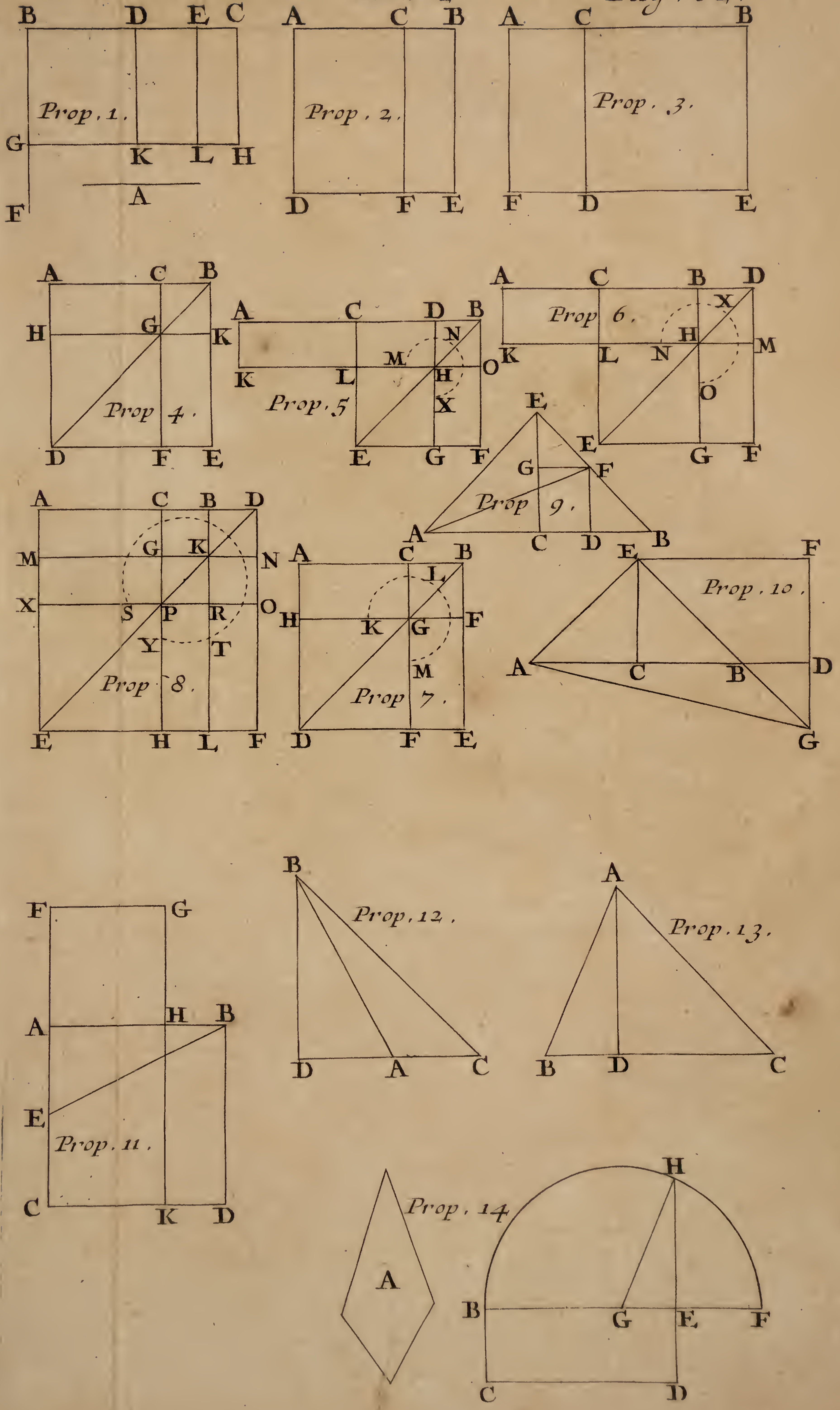
† 10. 1.

‡ 5 *of this.*

* 47. 1.

mon, be taken from both, the remaining Rectangle contained under BE and EF, is equal to the Square of EH. But the Rectangle under BE and EF is the Parallelogram BD, becauſe EF is equal to ED. Therefore the Parallelogram BD is equal to the Square of EH; but the Parallelogram BD is equal to the Right-lin'd Figure A. Wherefore the Right-lined Figure A is equal to the Square of EH. And ſo there is a Square made equal to the given Right-lined Figure A, *viz.* the Square of EH; *which was to be done.*

The END *of the* SECOND BOOK.


Prop. 1.
B D E C G K L H F A
Prop. 2.
A C B D F E
Prop. 3.
A C B F D E
Prop 4.
A C B H G K D F E
Prop. 5
A C D B N M O K L H X E G F
Prop 6.
A C B D X K L N H M O E G F
Prop 9.
E G F A C D B
Prop 8.
A C B D M G K N X S P R O Y T E H L F
Prop 7.
A C B L H K G F M D F E
Prop. 10.
E F A C B D G
Prop. 11.
F G A H B E C K D
Prop. 12.
B D A C
Prop. 13.
A B D C
Prop. 14
A
H B G E F C D

EUCLID's

ELEMENTS.

BOOK III.

DEFINITIONS.

I. *EQUAL Circles are ſuch whoſe Diameters are equal; or from whoſe Centres the Right Lines that are drawn are equal.*

II. *A Right Line is ſaid to touch a Circle, when touching the ſame, and being produced, does not cut it.*

III. *Circles are ſaid to touch each other, which touching do not cut one another.*

IV. *Right Lines in a Circle are ſaid to be equally diſtant from the Centre, when Perpendiculars drawn from the Centre to them be equal.*

V. *And that Line is ſaid to be farther from the Centre, on which the greater Perpendicular falls.*

VI. *A Segment of a Circle is a Figure contained under a Right Line, and a Part of the Circumference of a Circle.*

VII. *An Angle of a Segment is that which is contained by a Right Line, and the Circumference of a Circle.*

VIII. *An Angle is ſaid to be in a Segment, when ſome Point is taken in the Circumference thereof, and from it Right Lines are drawn to the Ends of that Right Line, which is the Baſe of the Segment; then the Angle contained under the Lines drawn, is ſaid to be an Angle in a Segment.*

IX. *But when the Right Lines containing the Angle do receive any Circumference of the Circle, then the Angle is ſaid to ſtand upon that Circumference.*

X. *A Sector of a Circle, is that Figure comprehended between the Right Lines drawn from the Centre, and the Circumference contained between them.*

XI. *Similar Segments of Circles are thoſe which include equal Angles, or whereof the Angles in them are equal.*

PROPOSITION I.

PROBLEM.

To find the Centre of a Circle given.

LET ABC be the Circle given. It is required to find the Centre thereof.

Let the Right Line AB be any how drawn
* 10. 1. in it, which * biſect in the Point D; and let
† 11. 1. DC be † drawn from the Point D at Right Angles to AB, which let be produced to E.

Then, if EC be * biſected in F, I ſay, the Point F is the Centre of the Circle ABC.

For, if it be not, let G be the Centre, and let GA, GD, GB, be drawn. Now, becauſe DA is equal to DB, and DG is common, the two Sides AD, DG, are equal to the two Sides GD, DB, each to each;
‡ *Def.* 15. 1. and the Baſe GA is ‡ equal to the Baſe GB; for they are drawn from the Centre G. Therefore the Angle
* 8. 1. ADG is * equal to the Angle GDB. But when a Right Line ſtanding upon a Right Line, makes the adjacent Angles equal to one another, each of the
† *Def.* 10. 1. equal Angles will † be a Right Angle. Wherefore the Angle GDB is a Right Angle. But FDB is alſo a

Right Angle. Therefore the Angle FDB is equal to the Angle GDB, a greater to a leſs, which is abſurd. Wherefore G is not the Centre of the Circle ABC. After the ſame manner we prove, that no other Point, unleſs F, is the Centre. Therefore F is the Centre of the Circle ABC; *which was to be found.*

Coroll. If in a Circle, any Right Line cuts another Right Line into two equal Parts, and at Right Angles; the Centre of the Circle will be in that cutting Line.

PROPOSITION II.

THEOREM.

If any two Points be aſſumed in the Circumference of a Circle, the Right Line joining thoſe two Points ſhall fall within the Circle.

LET ABC be a Circle; in the Circumference of which let any two Points A, B, be aſſumed. I ſay, a Right Line drawn from the Point A to the Point B, falls within the Circle.

For let any Point E be taken in the Right Line AB, and let DA, DE, DB, be joined.

Then, becauſe DA is equal to DB, the Angle DAB will be *equal to the Angle DBA; and ſince the Side AE of the Triangle DAE is produced, the Angle DEB will be † greater than the Angle DAE. But the Angle DAE is equal to the Angle DBE; therefore the Angle DEB is greater than the Angle DBE. But the greater Side ſubtends the greater Angle. Wherefore DB is greater than DE. But DB only comes to the Circumference of the Circle; therefore DE does not reach ſo far. And ſo the Point E falls within the Circle. Therefore, *if two Points are aſſumed in the Circumference of a Circle, the Right Line joining thoſe two Points ſhall fall within the Circle;* which was to be demonſtrated.

* 5. 1.
† 16. 1.

Coroll. Hence, if a Right Line touches a Circle, it will touch it in one Point only.

PROPOSITION III.

THEOREM.

If in a Circle a Right Line drawn thro' the Centre, cuts any other Right Line not drawn thro' the Centre, into equal Parts, it ſhall cut it at Right Angles; and if it cuts it at Right Angles, it ſhall cut it into two equal Parts.

LET ABC be a Circle, wherein the Right Line CD, drawn thro' the Centre, biſects the Right Line AB not drawn thro' the Centre. I ſay, it cuts it at Right Angles.

* 1 *of this.* For, *find E the Centre of the Circle, and let EA, EB, be joined.

Then, becauſe AF is equal to FB, and FE is common, the two Sides AF, FE, are equal to the two Sides BF, FE, each to each; but the Baſe EA is equal to the Baſe EB. Wherefore the Angle AFE
† 8. 1. ſhall be † equal to the Angle BFE. But when a Right Line ſtanding upon a Right Line makes the adjacent Angles equal to one another, each of the equal Angles
‡ *Def.* 10. 1. is ‡ a Right Angle. Wherefore AFE, or BFE, is a Right Angle. And therefore the Right Line CD drawn thro' the Centre, biſecting the Right Line AB not drawn thro' the Centre, cuts it at Right Angles. Now if CD cuts AB at Right Angles, I ſay, it will biſect it, that is, AF will be equal to FB. For the ſame Conſtruction remaining, becauſe EA, being drawn from the Centre, is equal to EB, the Angle EAF
* 5. 1. ſhall be *equal to the Angle EBF. But the Right Angle AFE is equal to the Right Angle BFE; therefore the two Triangles EAF, EBF, have two Angles of the one equal to two Angles of the other, and the Side EF is common to both. Wherefore the other Sides of
† 26. 1. the one ſhall be † equal to the other Sides of the other: And ſo AF will be equal to FB. Therefore, *if in a Circle a Right Line drawn thro' the Centre, cuts any other Right Line not drawn thro' the Centre, into two equal Parts, it ſhall cut it at Right Angles; and if it cuts it at Right Angles, it ſhall cut it into two equal Parts;* which was to be demonſtrated.

PROPOSITION IV.

THEOREM.

If in a Circle two Right Lines, not being drawn thro' the Centre, cut each other, they will not cut each other into two equal Parts.

LET ABCD be a Circle, wherein two Right Lines AC, BD, not drawn thro' the Centre, cut each other in the Point E. I ſay, they do not biſect each other.

For, if poſſible, let them biſect each other, ſo that AE be equal to EC, and BE to ED. Let the Centre F of the Circle ABCD be * found, and join EF. * *1 of this.*

Then, becauſe the Right Line FE, drawn thro' the Centre, biſects the Right Line AC, not drawn thro' the Centre, it will † cut AC at Right Angles. And † *3 of this.* ſo FEA is a Right Angle. Again, becauſe the Right Line FE biſects the Right Line BD not drawn thro' the Centre, it will † cut BD at Right Angles. Therefore FEB is a Right Angle. But FEA has been ſhewn to be alſo a Right Angle. Wherefore the Angle FEA will be equal to the Angle FEB, a leſs to a greater; which is abſurd. Therefore AC, BD, do not mutually biſect each other. And ſo, *if in a Circle two Right Lines, not being drawn thro' the Centre, cut each other, they will not cut each other into two equal Parts*; which was to be demonſtrated.

PROPOSITION V.

THEOREM.

If two Circles cut one another, they ſhall not have the ſame Centre.

LET the two Circles ABC, CDG, cut each other in the Points B, C. I ſay, they have not the ſame Centre.

For if they have, let it be E, and join EC, and draw EFG at Pleaſure.

Now, becauſe E is the Centre of the Circle ABC, CE will be equal to EF. Again, becauſe E is the Centre of the Circle CDG, CE is equal to EG. But CE has been ſhewn to be equal to EF. Therefore EF ſhall be equal to EG, a leſs to a greater, which cannot be. Therefore the Point E is not the Centre of both the Circles ABC, CDG. Wherefore, *if two Circles cut one another, they ſhall not have the ſame Centre*; which was to be demonſtrated.

PROPOSITION VI.

THEOREM.

If two Circles touch one another inwardly, they will not have one and the ſame Centre.

LET two Circles ABC, CDE, touch one another inwardly in the Point C. I ſay, they will not have one and the ſame Centre.

For if they have, let it be F, and join FC, and draw FB any how.

Then, becauſe F is the Centre of the Circle ABC, CF is equal to FB. And becauſe F is only the Centre of the Circle CDE, CF ſhall be equal to EF. But CF has been ſhewn to be equal to FB. Therefore FE is equal to FB, a leſs to a greater; which cannot be. Therefore the Point F is not the Centre of both the Circles ABC, CDE. Wherefore, *if two Circles touch one another inwardly, they will not have one and the ſame Centre*; which was to be demonſtrated.

PROPOSITION VII.

THEOREM.

If in the Diameter of a Circle ſome Point be taken, which is not the Centre of the Circle, and from that Point certain Right Lines fall on the Circumference of the Circle, the greateſt of theſe Lines ſhall be that wherein the Centre of the Circle is; the leaſt, the Remainder of the ſame Line. And of all the other Lines, the neareſt to that which was drawn thro' the Centre, is always greater than that more remote, and only two equal Lines fall from the abovеſaid Point upon the Circumference, on each Side of the leaſt or greateſt Lines.

LET ABCD be a Circle, whoſe Diameter is AD, in which aſſume ſome Point F, which is not the Centre of the Circle. Let the Centre of the Circle be E; and from the Point F let certain Right Lines FB, FC, FG, fall on the Circumference: I ſay, FA is the greateſt of theſe Lines, and FD the leaſt; and of the others FB is greater than FC, and FC greater than FG.

For let BE, CE, GE, be joined.

Then, becauſe two Sides of every Triangle are *greater than the third; BE, EF, are greater than BF. But AE is equal to BE. Therefore BE and EF are equal to AF. And ſo AF is greater than FB. * 20. 1.

Again, becauſe BE is equal to CE, and FE is common, the two Sides BE and FE are equal to the two Sides CE, EF. But the Angle BEF is greater than the Angle CEF. Wherefore the Baſe BF is greater than the Baſe FC†. For the ſame Reaſon, CF is greater than FG. † 24. 1.

Again, becauſe GF and FE are *greater than GE, and GE is equal to ED; GF and FE ſhall be greater than ED; and if FE, which is common, be taken away, then the Remainder GF is greater than the Remainder FD. Wherefore FA is the * 20. 1.

greateſt of the Right Lines, and FD the leaſt: Alſo BF is greater than FC, and FC greater than FG.

I ſay, moreover, that there are only two equal Right Lines, that can fall from the Point F on ABCD, the Circumference of the Circle on each Side the ſhorteſt Line FD. For at the given Point E, ‡ 23. 1. with the Right Line EF, make ‡ the Angle FEH equal to the Angle GEF, and join FH. Now becauſe GE is equal to EH, and EF is common, the two Sides GE and EF are equal to the two Sides HE and EF. But the Angle GEF is equal to the Angle HEF. Therefore the Baſe FG ſhall † 4. 1. be † equal to the Baſe FH. I ſay, no other Right Line falling from the Point F, on the Circle, can be equal to FG. For if there can, let this be FK. Now ſince FK is equal to FG, as alſo FH, FK will be equal to FH, *viz.* a Line drawn nigher to that paſſing thro' the Centre, equal to one more remote, which cannot be. *If, therefore, in the Diameter of a Circle, ſome Point be taken, which is not the Centre of the Circle, and from that Point certain Right Lines fall on the Circumference of the Circle, the greateſt of theſe Lines ſhall be that wherein the Centre of the Circle is; the leaſt, the Remainder of the ſame Line. And of all the other Lines, the neareſt to that which was drawn through the Centre, is always greater than that more remote; and only two equal Lines fall from the aboveſaid Point upon the Circumference, on each Side of the leaſt or greateſt Lines;* which was to be demonſtrated.

PRO-

PROPOSITION VIII.

THEOREM.

If ſome Point be aſſumed without a Circle, and from it certain Right Lines be drawn to the Circle, one of which paſſes thro' the Centre, but the other any how; the greateſt of theſe Lines, is that paſſing thro' the Centre, and falling upon the Concave Part of the Circumference of the Circle; and of the others, that which is neareſt to the Line paſſing thro' the Centre is greater than that more remote. But the leaſt of the Lines that fall upon the Convex Circumference of the Circle, is that which lies between the Point and the Diameter; and of the others, that which is nigher to the leaſt, is leſs than that which is further diſtant; and from that Point there can be drawn only two equal Lines, which ſhall fall on the Circumference on each Side the leaſt Line.

LET ABC be a Circle, out of which take any Point D. From this Point let there be drawn certain Right Lines DA, DE, DF, DC, to the Circle, whereof DA paſſes thro' the Centre. I ſay DA, which paſſes thro' the Centre, is the greateſt of the Lines falling upon AEFC, the Concave Circumference of the Circle; and the leaſt is DG, *viz.* the Line drawn from D to the Diameter GA: Likewiſe DE is greater then DF, and DF greater than DC. But of theſe Lines that fall upon HLGK the Convex Circumference of the Circle, that which is neareſt the leaſt DG, is always leſs than that more remote; that is, DK is leſs than DL, and DL leſs than DH.

For, find * M the Centre of the Circle ABC, and let ME, MF, MC, MH, ML, be joined. * 1 *of this.*

Now, becauſe AM is equal to ME; if MD, which is common, be added, AD will be equal to EM and MD. But EM and MD are † greater than ED; therefore AD is alſo greater than ED. † 20.

Again, becauſe ME is equal to MF, and MD is common, then EM, MD, ſhall be equal to MF, MD; and the Angle EMD is greater than the Angle FMD. Therefore the Baſe ED will be † greater than the Baſe FD. We prove, in the ſame manner, that FD is greater than CD. Wherefore DA is the greateſt of the Right Lines falling from the Point D; DE is greater than DF, and DF is greater than DC.

† 24. 1.

Moreover, becauſe MK and KD are * greater than MD, and MG is equal to MK; then the Remainder KD will † be greater than the Remainder GD. And ſo GD is leſs than KD, and conſequently is the leaſt. And becauſe two Right Lines MK, KD, are drawn from M and D to the Point K, within the Triangle MLD, MK, and KD, are ‡ leſs than ML and LD; but MK is equal to ML. Wherefore the Remainder DK is leſs than the Remainder DL. In like manner we demonſtrate, that DL is leſs than DH. Therefore DG is the leaſt. And DK is leſs than DL, and DL than DH.

* 20. 1.
† *Ax.* 4.
‡ 21. 1.

I ſay, likewiſe, that from the Point D only two equal Right Lines can fall upon the Circle on each Side the leaſt Line. For, make * the Angle DMB at the Point M, with the Right Line MD, equal to the Angle KMD, and join DB. Then, becauſe MK is equal to MB, and MD is common, the two Sides KM, MD, are equal to the two Sides BM, MD, each to each; but the Angle KMD is equal to the Angle BMD. Therefore the Baſe DK is † equal to the Baſe DB. I ſay, no other Line can be drawn from the Point D to the Circle equal to DK; for if there can, let DN. Now, ſince DK is equal to DN, as alſo to DB, therefore DB ſhall be equal to DN, *viz.* the Line drawn neareſt to the leaſt equal to that more remote, which has been ſhewn to be impoſſible. Therefore, *if ſome Point be aſſumed without a Circle, and from it certain Right Lines be drawn to the Circle, one of which paſſes through the Centre, but the others any how; the greateſt of theſe Lines is that paſſing through the Centre, and falling upon the Concave Part of the Circumference of the Circle; and of the others, that which is neareſt to the Line paſſing through the Centre, is greater than that more*

* 23. 1.
† 4. 1.

more remote. But the leaſt of the Lines that fall upon the Convex Circumference of the Circle, is that which lies between the Point and the Diameter; and of the others, that which is nigher to the leaſt, is leſs than that which is farther diſtant; and from that Point there can be drawn only two equal Lines, which ſhall fall on the Circumference on each Side the leaſt Line; which was to be demonſtrated.

PROPOSITION IX.

THEOREM.

If a Point be aſſumed in a Circle, and from it more than two equal Right Lines be drawn to the Circumference; then that Point is the Centre of the Circle.

LET the Point D be aſſumed within the Circle ABC; and from the Point D, let there fall more than two equal Right Lines to the Circumference, *viz.* the Right Lines DA, DB, DC. I ſay, the aſſumed Point D is the Centre of the Circle ABC.

For, if it be not, let E be the Centre, if poſſible; and join DE, which produce to G and F.

Then FG is a Diameter of the Circle ABC; and ſo, becauſe the Point D, not being the Centre of the Circle, is aſſumed in the Diameter FG, DG will * be the greateſt Line drawn from D to the Circumference, and DC greater than DB, and DB than DA; but they are alſo equal, which is abſurd. Therefore E is not the Centre of the Circle ABC. And in this manner we prove, that no other Point except D is the Centre; therefore D is the Centre of the Circle ABC; *which was to be demonſtrated.*

* 7 *of this.*

Otherwiſe:

Let ABC be the Circle, within which take the Point D, from which let more than two equal Right Lines fall on the Circumference of the Circle, *viz.* the three equal ones DA, DB, DC: I ſay, the Point D is the Centre of the Circle ABC.

For,

*10. 1. Euc. For, join A B, B C, which biſect * in the Points E and Z; as alſo join E D, D Z; which produce to the Points H, K, O, L; then, becauſe A E is equal to E B, and E D is common, the two Sides A E, E D, ſhall be equal to the two Sides B E, E D. And the Baſe D A is equal to the Baſe Baſe D B: Therefore the Angle A E D will be * equal to the Angle B E D: *8. 1. and ſo [by *Def.* 10. 1.] each of the Angles A E D, B E D, is a Right Angle: Therefore H K, biſecting A B, cuts it at Right Angles. And becauſe a Right Line in a Circle, biſecting another Right Line, cuts it at Right Angles, and the Centre of the Circle is in the cutting Line, [by *Cor.* 1. 3.] the Centre of the Circle A B C will be in H K. For the ſame Reaſon, the Centre of the Circle will be in O L. And the Right Lines H K, O L, have no other Point common but D: Therefore D is the Centre of the Circle A B C; *which was to be demonſtrated.*

PROPOSITION X.

THEOREM.

A Circle cannot cut another Circle in more than two Points.

FOR if it can, let the Circle A B C cut the Circle D E F in more than two Points, *viz.* in B, G, F; and let K be the Centre of the Circle A B C, and join K B, K G, K F.

Now, becauſe the Point K is aſſumed within the Circle D E F, from which more than two equal Right Lines K B, K G, K F, fall on the Circumference, † 9 *of this.* the Point K ſhall be † the Centre of the Circle D E F. ‡ *By Hyp.* But K is ‡ the Centre of the Circle A B C. Therefore K will be the Centre of two Circles cutting each other, which is abſurd. Wherefore a Circle cannot cut a Circle in more than two Points; *which was to be demonſtrated.*

PRO-

PROPOSITION XI.

THEOREM.

If two Circles touch each other on the Inſide, and the Centres be found, the Line joining their Centres, will fall on the [Point of] Contact of thoſe Circles.

LET two Circles ABC, ADE, touch one another inwardly in A, and let F be the Centre of the Circle ABC, and G that of ADE. I ſay, a Right Line joining the Centres G and F, being produced, will fall in the Point A.

If this be denied, let the Right Line, joining FG, cut the Circle in D and H.

Now, becauſe AG, GF, are greater than AF, * that is, than FH; take away FG, which is common, and the Remainder AG is greater than the Remainder GH. But AG is equal to GD; therefore GD is greater than GH, the leſs than the greater, which is abſurd. Wherefore a Line drawn thro' the Points F, G, will not fall out of the Point of Contact A, and ſo neceſſarily muſt fall in it; *which was to be demonſtrated.* * 20. 1.

PROPOSITION XII.

THEOREM.

If two Circles touch one another on the Outſide, a Right Line joining their Centres will paſs thro' the [Point of] Contact.

LET two Circles ABC, ADE, touch one another outwardly in the Point A; and let F be the Centre of the Circle ABC, and G that of ADE. I ſay, a Right Line drawn through the Centre F, G, will paſs thro' the Point of Contact A.

For, if it does not, let, if poſſible, FCDG fall without it, and join FA, AG.

Now, ſince F is the Centre of the Circle ABC, AF will be equal to FC. And becauſe G is the Centre of the Circle A D E, A G will be equal to G D: But A F has been ſhewn to be equal to F C; therefore F A, A G, are equal to F C, D G. And ſo the Whole F G is greater than F A, A G; and alſo leſs, * which is abſurd. Therefore a Right Line, drawn from the Point F to G, will paſs through the Point of Contact A; *which was to be demonſtrated.*

* 20. 1.

PROPOSITION XIII.

THEOREM.

One Circle cannot touch another in more Points than one, whether it be inwardly or outwardly.

FOR, in the firſt Place, if this be denied, let the Circle A B D C, if poſſible, touch the Circle E B F D inwardly, in more Points than one, *viz.* in B D.

And let G be the Centre of the Circle A B D C, and H that of E B F D.

Then a Right Line, drawn from the Point G to H, † 11 *of this.* will † fall in the Points B and D. Let this Line be B G H D. And becauſe G is the Centre of the Circle A B D C, the Line B G will be equal to G D. Therefore B G is greater than H D, and B H much greater than H D. Again, ſince H is the Centre of the Circle E B F D, the Line B H is equal to H D. But it has been proved to be much greater than it, which is abſurd. Therefore one Circle cannot touch another Circle inwardly in more Points than one.

Secondly, let the Circle A C K, if poſſible, touch the Circle A B D C outwardly in more Points than one, *viz.* in A and C; and let A, C, be joined.

Now, becauſe two Points A, C, are aſſumed in the Circumference of each of the Circles A B D C, A C K, ‡ 2 *of this.* a Right Line joining theſe two Points, will fall ‡ within either of the Circles. But it falls within the Circle A B D C, and without the Circle A C K, which is abſurd. Therefore one Circle cannot touch another Circle in more Points than one outwardly. But it has been proved, that one Circle cannot touch another Circle inwardly [in more Points than one]. Wherefore *one Circle cannot touch another in more*

Points than one, whether it be inwardly or outwardly; which was to be demonſtrated.

PROPOSITION XIV.

THEOREM.

Equal Right Lines in a Circle are equally diſtant from the Centre; and Right Lines, which are equally diſtant from the Centre, are equal between themſelves.

LET ABDC be a Circle, wherein are the equal Right Lines AB, CD. I ſay, theſe Lines are equally diſtant from the Centre of the Circle.

For, let E be the Centre of the Circle ABDC; from which let there be drawn EF, EG, perpendicular to AB, CD; and let AE, EC, be joined.

Then, becauſe a Right Line EF, drawn thro' the Centre, cuts the Right Line AB, not drawn thro' the Centre, at Right Angles, it will * biſect the ſame. * 3 *of this.* Wherefore AF is equal to FB, and ſo AB is double to AF. For the ſame Reaſon CD is double to CG, but AB is equal to CD. Therefore AF is equal to CG; and becauſe AE is equal to EC, the Square of AE ſhall be equal to the Square of EC. But the Squares of AF and FE are † equal to the Square of † 47. 1. AE. For the Angle at F is a Right Angle; and the Squares of EG, and GC, are equal to the Square of EC, ſince the Angle at G is a Right one. Therefore the Squares of AF and FE, are equal to the Squares of CG and GE. But the Square of AF is equal to the Square of CG; for AF is equal to CG. Therefore the Square of FE is equal to the Square of EG; and ſo FE equal to EG. Alſo Lines in a Circle are ‡ ſaid to be equally diſtant from ‡ *Def.* 4. *of this.* the Centre, when Perpendiculars drawn to them from the Centre are equal. Therefore AB, CD, are equally diſtant from the Centre.

But if AB, CD, are equally diſtant from the Centre, that is, if FE be equal to EG; I ſay, AB is equal to CD.

For the ſame Conſtruction being ſuppoſed, we demonſtrate as above, that AB is double to AF, and

CD to CG; and becauſe AE is equal to EC, the Square of AE will be equal to the Square of EC. But the Squares of EF and FA are † equal to the Square of AE, and the Squares of EG and GC equal † to the Square of EC. Therefore the Squares of EF and FA are equal to the Squares of EG and GC. But the Square of EG is equal to the Square of EF; for EG is equal to EF. Therefore the Square of AF is equal to the Square of CG; and ſo AF is equal to CG. But AB is double to AF, and CD to CG. Therefore *equal Right Lines in a Circle are equally diſtant from the Centre; and Right Lines, which are equally diſtant from the Centre, are equal between themſelves*; which was to be demonſtrated. † 47. 1.

PROPOSITION XV.

THEOREM.

A Diameter is the greateſt Line in a Circle; and of all the other Lines therein, that which is neareſt to the Centre is greater than that more remote.

LET ABCD be a Circle, whoſe Diameter is AD, and Centre E; and let BC be nearer to the Diameter than FG. I ſay, AD is the greateſt, and BC is greater than FG.

For, let the Perpendiculars EH, EK, be drawn from the Centre E to BC, FG. Now, becauſe BC is nearer to the Centre than FG, EK will be greater than EH. Let EL be equal to EH; draw LM through L at Right Angles to EK, which produce to N; and let EM, EN, EF, EG, be joined.

Then, becauſe EH is equal to EL, the Line BC will be equal to MN*. And, ſince AE is equal to EM, and DE to EN, AD will be equal to ME and EN. But ME and EN are † greater than MN: And ſo AD is greater than MN; and NM is equal to BC. Therefore AD is greater than BC. And ſince the two Sides EM, EN, are equal to the two Sides FE, EG, and the Angle MEN greater than the Angle FEG, the Baſe MN ſhall be ‡ greater than the Baſe FG. But MN is equal to BC. Therefore

* 14 *of this.*

† 20. 1.

‡ 24. 1.

fore B C is greater than F G. And ſo the Diameter A D is the greateſt, and B C is greater than F G. Wherefore *the Diameter is the greateſt Line in a Circle; and of all the other Lines therein, that which is neareſt to the Centre is greater than that more remote;* which was to be demonſtrated.

PROPOSITION XVI.

THEOREM.

A Line drawn from the extreme [Point] of the Diameter of a Circle at Right Angles to that Diameter, ſhall fall without the Circle; and between the ſaid Right Line, and the Circumference, no other Line can be drawn; and the Angle of a Semicircle is greater than any Right-lin'd acute Angle; and the remaining Angle [without any Circumference] is leſs than any Right-lined Angle.

LET A B C be a Circle, whoſe Centre is D, and Diameter A B. I ſay, a Right Line drawn from the Point A at Right Angles to A B, falls without the Circle.

For if it does not, let it fall, if poſſible, within the Circle, as A C, and join D C.

Now, becauſe D A is equal to D C, the Angle D A C ſhall be * equal to the Angle A C D. But D A C is a Right Angle; therefore A C D is a Right Angle: And accordingly the Angles D A C, A C D, are equal to two Right Angles; which is abſurd †. Therefore a Right Line drawn from the Point A at Right Angles to B A, will not fall within the Circle; and ſo likewiſe we prove, that it neither falls in the Circumference. Therefore it will neceſſarily fall without the ſame; which now let be A E.

* 5. 1.

† 17. 1.

Again, between the Right Line A E, and the Circumference C H A, no other Right Line can be drawn.

For if there can, let it be F A, and let ‡ D G be drawn at Right Angles from the Centre D to F A.

‡ 12. 1.

Now, becauſe A G D is a Right Angle, and D A G is leſs than a Right Angle, D A will be greater than D G *. But D A is equal to D H. Therefore D H is greater

* 19. 1.

greater than D G, the leſs than the greater; which is abſurd. Wherefore no Right Line can be drawn between A E, and the Circumference A H C. I ſay, moreover, that the Angle of the Semicircle, contained under the Right Line B A, and the Circumference C H A, is greater than any Right-lined acute Angle; and the remaining Angle contained under the Circumference C H A, and the Right Line A E, is leſs than any Right-lined Angle.

For if any Right-lined acute Angle be greater than the Angle contained under the Right Line B A, and the Circumference C H A; or if any Right-lined Angle be leſs than that contained under the Circumference C H A, and the Right Line A E, then a Right Line may be drawn between the Circumference C H A and the Right Line A E, making an Angle greater than that contained under the Right Line B A, and the Circumference C H A, *viz.* which is contained under Right Lines, and leſs than that contained under the Circumference C H A, and the Right Line A E. But ſuch a Right Line cannot be drawn from what has been proved. Therefore no Right-lined acute Angle is greater than the Angle contained under the Right Line B A, and the Circumference C H A; nor leſs than the Angle contained under the Circumference C H A, and the Right Line A E.

Coroll. From hence it is manifeſt, that a Right Line drawn at Right Angles on the End of the Diameter of a Circle, touches the Circle, and that in one Point only, becauſe, if it ſhould meet it in two Points, it would fall within the ſame; *as has been demonſtrated.*

PROPOSITION XVII.

PROBLEM.

To draw a Right Line from a given Point, that ſhall touch a given Circle.

LET A be the Point given, and B C D the Circle. It is required to draw a Right Line from the Point A, that ſhall touch the given Circle B C D.

Let E be the Centre of the Circle, and join AE; then about the Centre E, with the Diſtance EA, deſcribe the Circle AFG; draw DF * at Right Angles to EA, and join EBF, and AB. I ſay, the Right Line AB is drawn from the Point A, touching the Circle BCD. * 11. 1.

For, ſince E is the Centre of the Circles BCD, AFG, the Line EA will be equal to EF, and ED to EB. Therefore the two Sides AE, EB, are equal to the two Sides FE, ED, each to each; and they contain the common Angle E. Wherefore the Baſe DF is † equal to the Baſe AB, and the Triangle DEF equal to the Triangle EBA, and the remaining Angles of the one equal to the remaining Angles of the other. And ſo the Angle EBA is equal to the Angle EDF. But EDF is a Right Angle. Wherefore EBA is alſo a Right Angle, and EB is a Line drawn from the Centre; but a Right Line drawn from the Extremity of the Diameter of a Circle at Right Angles ‡ to it, touches the Circle. Wherefore AB touches the Circle; *which was to be done.* † 4. 1. ‡ *Cor.* 16. *of this.*

PROPOSITION XVIII.

THEOREM.

If any Right Line touches a Circle, and from the Centre to the Point of Contact a Right Line be drawn; that Line will be perpendicular to the Tangent.

LET any Right Line DE touch a Circle ABC in the Point C, and let there be drawn the Right Line FC from the Centre FC. I ſay, FC is perpendicular to DE.

For, if it be not, let FG be drawn * from the Centre F, perpendicular to DE. * 12. 1.

Now, becauſe the Angle FGC is a Right Angle, the Angle GCF will be † an acute Angle; and accordingly the Angle FGC is greater than the Angle FCG; but the greater Side ſubtends ‡ the greater Angle. Therefore FC is greater than FG. But FC is equal to FB. Wherefore FB is greater than FG, a leſs than the greater; which is abſurd. Therefore † 32. 1. ‡ 19. 1.

FG is not perpendicular to DE. And in the ſame manner we prove, that no other Right Line but FC is perpendicular to DE. Wherefore FC is perpendicular to DE. Therefore, *if any Right Line touches a Circle, and from the Centre to the Point of Contact a Right Line be drawn; that Line will be perpendicular to the Tangent*; which was to be demonſtrated.

PROPOSITION XIX.

THEOREM.

If any Right Line touches a Circle, and from the Point of Contact a Right Line be drawn at Right Angles to the Tangent, the Centre of the Circle ſhall be in the ſaid Line.

LET any Right Line DE touch the Circle ABC in C, and let CA be drawn from the Point C at Right Angles to DE. I ſay, the Circle's Centre is in AC.

For if it be not, let F be the Centre, if poſſible; and join CF.

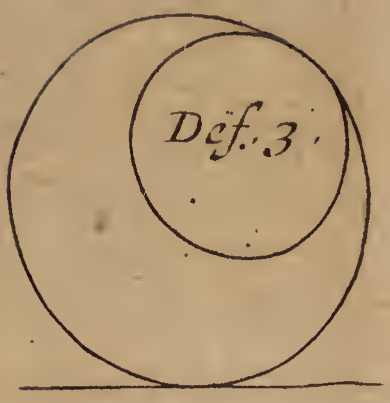
Then, becauſe the Right Line DE touches the Circle ABC, and FC is drawn from the Centre to the Point of Contact; FC will be perpendicular to DE *. And ſo the Angle FCE is a Right one. But ACE is alſo a Right Angle †: Therefore the Angle FCE is equal to the Angle ACE, a leſs to a greater; which is abſurd. Therefore F is not the Centre of the Circle ABC. After this manner we prove, that the Centre of the Circle can be in no other Line, unleſs in AC. Wherefore, *if any Right Line touches a Circle, and from the Point of Contact a Right Line be drawn at Right Angles to the Tangent, the Centre of the Circle ſhall be in the ſaid Line*; which was to be demonſtrated.

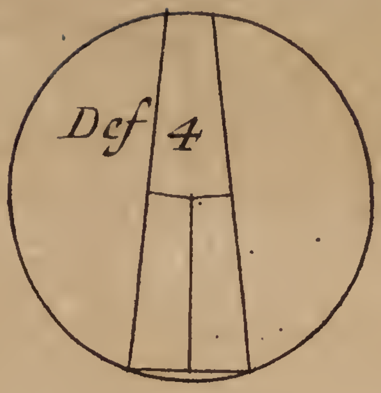
* 18 *of this.*

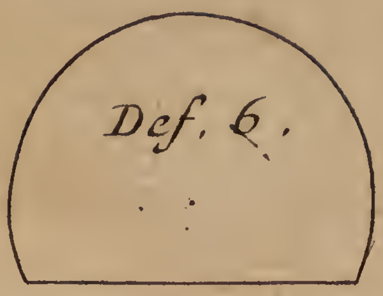
† *From the Hyp.*

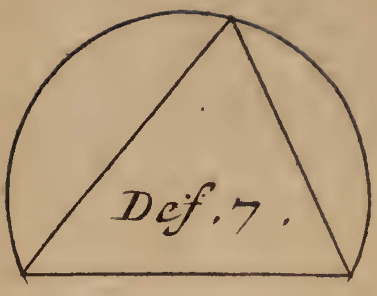
PRO-

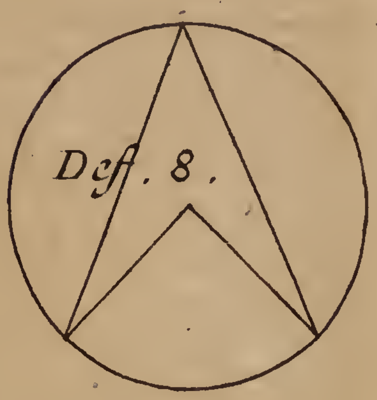
Book 3. Plate 1.

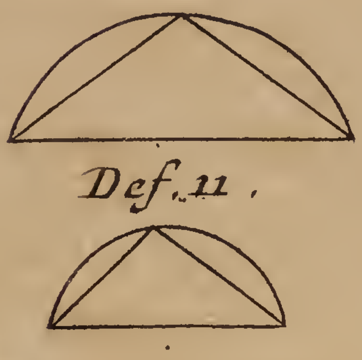


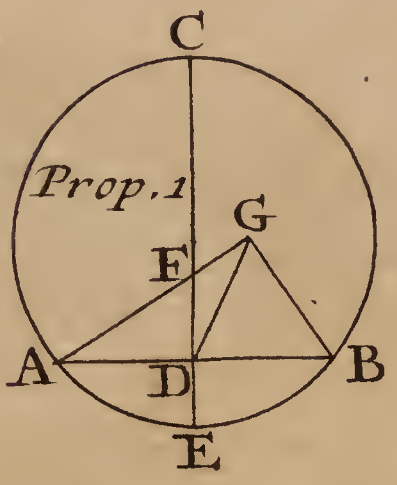


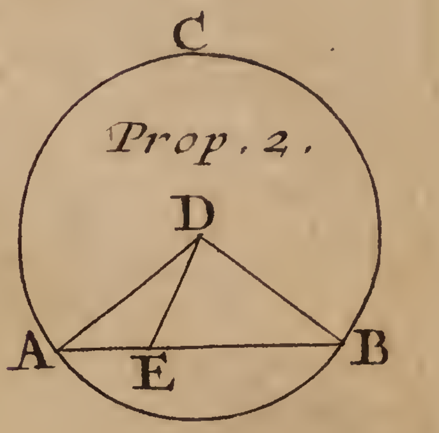


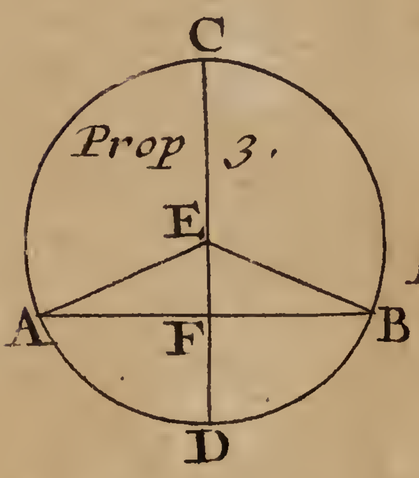


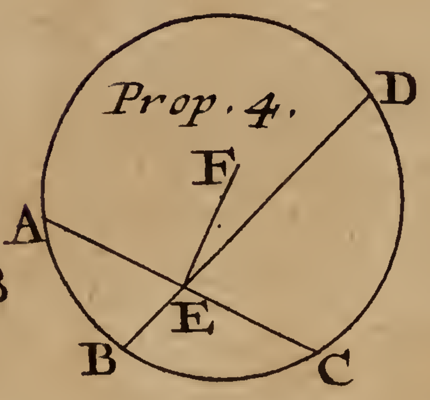


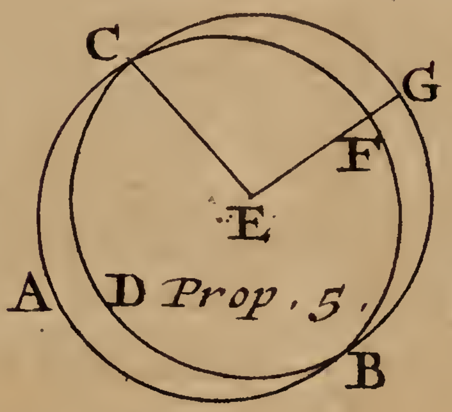


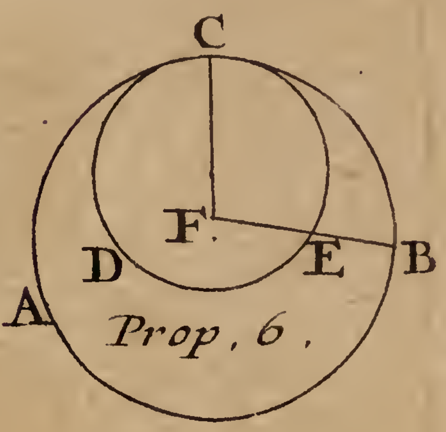


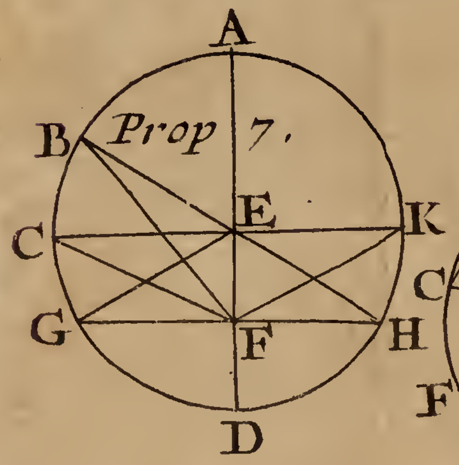


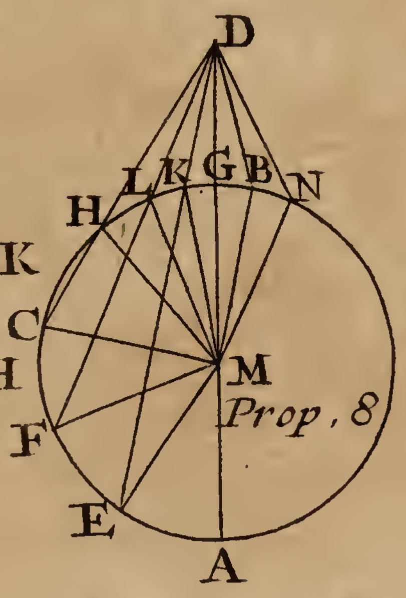


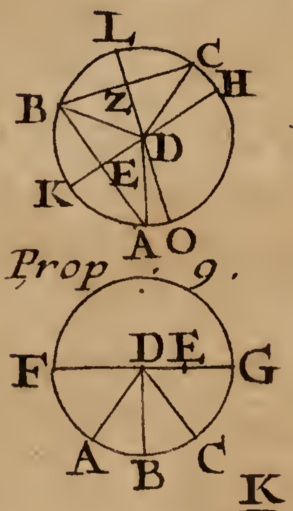


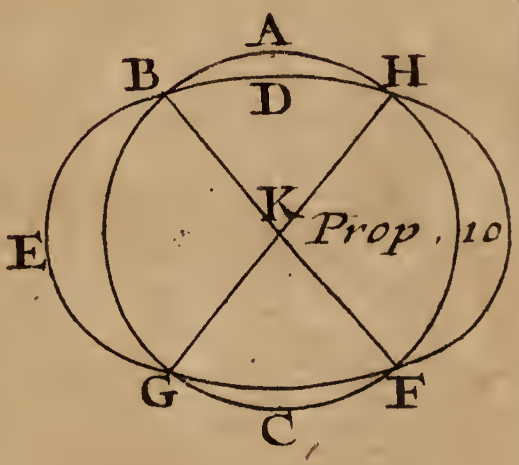


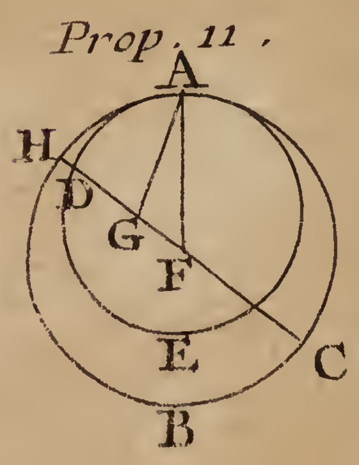


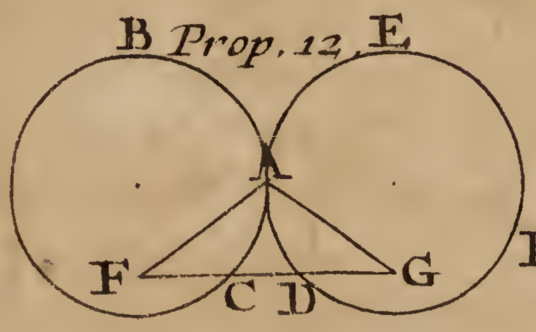


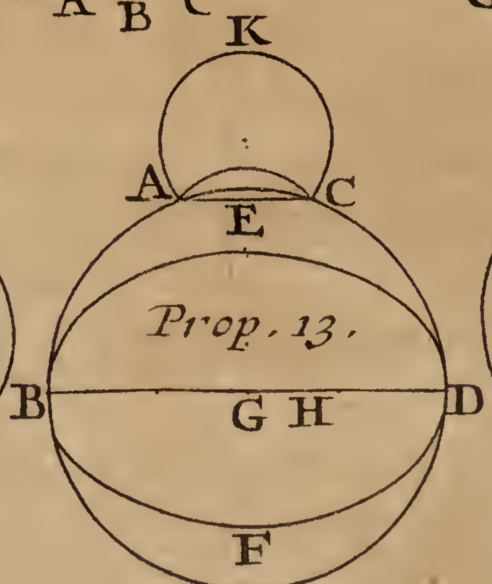


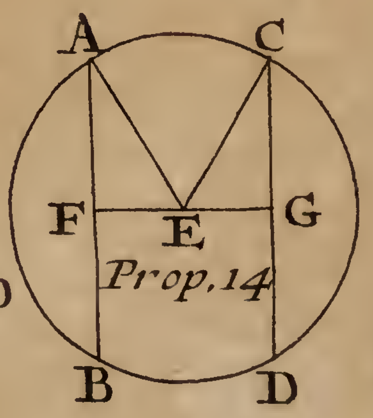


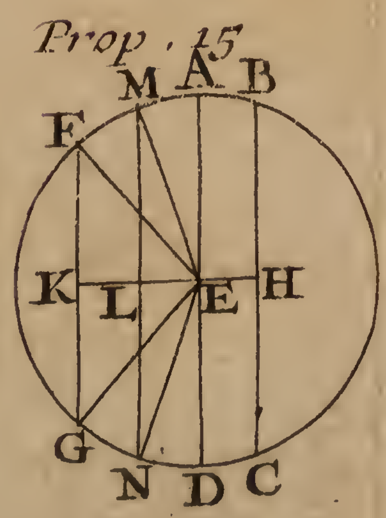


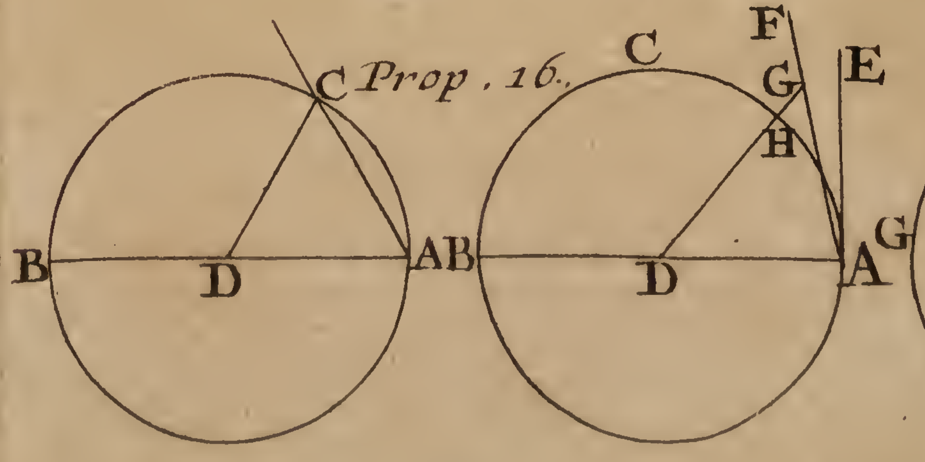


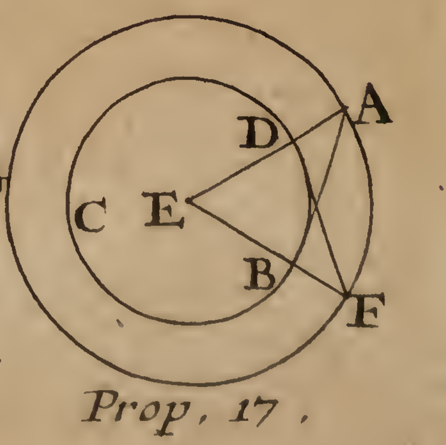












PROPOSITION XX.

THEOREM.

The Angle at the Centre of a Circle is double to the Angle at the Circumference, when the ſame Arc is the Baſe of the Angles.

LET ABC be a Circle, at the Centre whereof is the Angle BEC, and at the Circumference the Angle BAC, both of which ſtand upon the ſame Arc BC. I ſay, the Angle BEC is double to the Angle BAC.

For join AE, and produce it to F.

Then, becauſe EA is equal to EB, the Angle EAB ſhall be equal to the Angle EBA*. Therefore the Angles EAB, EBA, are double to the Angle EAB; but the Angle BEF is † equal to the Angles EAB, EBA; therefore the Angle BEF is double to the Angle EAB. For the ſame Reaſon, the Angle FEC is double to EAC. Therefore the whole Angle BEC is double to the whole Angle BAC. Again, let there be another Angle BDC, and join DE, which produce to G. We demonſtrate in the ſame manner, that the Angle GEC is double to the Angle GDC; whereof the Part GEB is double to the Part GDB. And therefore BEC is double to BDC. Conſequently, *an Angle at the Centre of a Circle is double to the Angle at the Circumference, when the ſame Arc is the Baſe of the Angles*; which was to be demonſtrated.

* 5. 1.

† 32. 1.

PROPOSITION XXI.

THEOREM.

Angles that are in the ſame Segment in a Circle, are equal to each other.

LET ABCDE be a Circle, and let BAD, BED, be Angles in the ſame Segment BAED. I ſay, thoſe Angles are equal.

For, let F be the Centre of the Circle ABCDE, and join BF, FD.

Now, becauſe the Angle BFD is at the Centre, and the Angle BAD at the Circumference, and they ſtand upon the ſame Arc BCD; the Angle BFD will be * double to the Angle BAD. For the ſame Reaſon, the Angle BFD is alſo double to the Angle BED. Therefore the Angle BAD will be equal to the Angle BED.

* 20 *of this.*

If the Angles BAD, BED, are in a Segment leſs than a Semiircle, let AE be drawn; and then all the Angles of the Triangle ABG are † equal to all the Angles of the Triangle DEG. But the Angles ABE, ADE, are equal, from what has been before proved, and the Angles AGB, DGE, are alſo equal ‡; for they are vertical Angles. Wherefore the remaining Angle BAG is equal to the remaining Angle GED. Therefor, *Angles that are in the ſame Segment in a Circle, are equal to each other;* which was to be demonſtrated.

† 32. 1.

‡ 15. 1.

PROPOSITION XXII.

THEOREM.

The oppoſite Angles of any quadrilateral Figure, deſcribed in a Circle, are equal to two Right Angles.

LET ABDC be a Circle, wherein is deſcribed the quadrilateral Figure ABCD. I ſay, two oppoſite Angles thereof are equal to two Right Angles.

For, join AD, BC.

Then, becauſe the three Angles of any Triangle are * equal to two Right Angles, the three Angles of the Triangle ABC, *viz.* the Angles CAB, ABC, BCA, are equal to two Right Angles. But the Angle ABC is † equal to the Angle ADC; for they are both in the ſame Segment ABDC. And the Angle ACB is † equal to the Angle ADB, becauſe they are in the ſame Segment ACDB; therefore the whole Angle BDC is equal to the Angles ABC, ACB; and if the common Angle BAC be added, then the Angles BAC, ABC, ACB, are equal to the Angles BAC, BDC; but the Angles BAC, ABC, ACB, are equal

* 32. 1.

† 21 *of this.*

* equal to the two Right Angles. Therefore likewiſe, * 32. 1. the Angles B A C, B D C, ſhall be equal to two Right Angles. And after the ſame way we prove, that the Angles A B D, A C D, are alſo equal to two Right Angles. Therefore *the oppoſite Angles of any quadrilateral Figure, deſcribed in a Circle, are equal to two Right Angles*; which was to be demonſtrated.

PROPOSITION XXIII.

THEOREM.

Two ſimilar and unequal Segments of two Circles cannot be ſet upon the ſame Right Line, and on the ſame Side thereof.

FOR if this be poſſible, let the two ſimilar and unequal Segments A C B, A D B, of two Circles ſtand upon the Right Line A B on the ſame Side thereof. Draw A C D, and let C B, B D, be joined. Now, becauſe the Segment A C B is ſimilar to the Segment A D B, and ſimilar Segments of Circles are * ſuch * *Def.* 11. *of this.* which receive equal Angles; the Angle A C B will be equal to the Angle A D B, the outward one to the inward one; which is † abſurd. Therefore *ſimilar* † 16. 1. *and unequal Segments of two Circles, cannot be ſet upon the ſame Right Line, and on the ſame Side thereof*; which was to be demonſtrated.

PROPOSITION XXIV.

THEOREM.

Similar Segments of Circles, being upon equal Right Lines, are equal to one another.

LET A E B, C F D, be equal Segments of Circles ſtanding upon the equal Right Lines A B, C D. I ſay, the Segment A E B is equal to the Segment C F D.

For the Segment A E B being applied to the Segment C F D, ſo that the Point A coincides with C, and the Line A B with C D; then the Point B will coincide with the Point D, ſince A B and C D are equal. And ſince the Right Line A B coincides with

CD, the Segment AEB will coincide with the Segment CFD. For if at the ſame time that AB coincides with CD, the Segment AEB ſhould not coincide with the Segment CFD, but be otherwiſe, as CGD; then a Circle would cut a Circle in more Points than two, *viz.* in the Points C, G, D; which
* *10 of this.* is * impoſſible. Wherefore, if the Right Line AB coincides with CD, the Segment AEB will coincide with, and be equal to, the Segment CFD. Therefore *ſimilar Segments of Circles, being upon equal Right Lines, are equal to one another*; which was to be demonſtrated.

PROPOSITION XXV.

PROBLEM.

A Segment of a Circle being given, to deſcribe the Circle whereof it is the Segment.

LET ABC be a Segment of a Circle given. It is required to deſcribe a Circle, whereof ABC is a Segment.

* 10. 1. Biſect * AC in D, and let DB be drawn † from
† 11. 1. the Point D at Right Angles to AC, and join AB. Now the Angle ABD is either greater, equal, or leſs than the Angle BAD. And firſt let it be greater, and
‡ 23. 1. make ‡ the Angle BAE at the given Point A, with the Right Line BA, equal to the Angle ABD; produce DB to E, and join EC.

Then, becauſe the Angle ABE is equal to the Angle
* 6. 1. BAE, the Right Line BE will be * equal to EA. And becauſe AD is equal to DC, and DE common, the two Sides AD, DE, are each equal to the two Sides CD, DE; and the Angle ADE is equal to the Angle CDE; for each is a Right one. Therefore the Baſe AE is equal to the Baſe EC. But AE has been proved to be equal to EB. Wherefore BE is alſo equal to EC. And accordingly the three Right Lines AE, EB, EC, are equal to each other. Therefore a Circle deſcribed about the Centre E, with either of the Diſtances AE, EB, EC, ſhall paſs thro' the other Points, and be that required to be deſcribed. But it is manifeſt, that the Segment ABC is leſs than a Semi-

a Semicircle, becauſe the Centre thereof is without the ſame.

But if the Angle ABD be equal to the Angle BAD; then if AD be made equal to BD, or DC, the three Right Lines AD, DB, DC, are equal between themſelves, and D will be the Centre of the Circle to be deſcribed; and the Segment ABC is a Semicircle.

But if the Angle ABD is leſs than the Angle BAD, let the Angle BAE be made, at the given Point A with the Right Line BA, within the Segment ABC, equal to the Angle ABD.

Then the Point E, in the Right Line DB, will be the Centre, and ABC a Segment greater than a Semicircle. Therefore a Circle is deſcribed, whereof a Segment is given; *which was to be done.*

PROPOSITION XXVI.

THEOREM.

In equal Circles, equal Angles ſtand upon equal Circumferences, whether they be at their Centres, or at their Circumferences.

LET ABC, DEF, be equal Circles, and let BGC, EHF, be equal Angles at their Centres, and, BAC, EDF, equal Angles at their Circumferences. I ſay, the Circumference BKC is equal to the Circumference ELF.

For, let BC, EF, be joined. Becauſe ABC, DEF, are equal Circles, the Lines drawn from their Centres will be equal. Therefore the two Sides BG, GC, are equal to the two Sides EH, HF; and the Angle G is equal to the Angle at H. Wherefore the Baſe BC is * equal to the Baſe EF. Again, becauſe the Angle at A is equal to that at D, the Segment BAC will be † ſimilar to the Segment EDF; and they are upon equal Right Lines BC, EF. But thoſe ſimilar Segments of Circles, that are upon equal Right Lines, are ‡ equal to each other. Therefore the Segment BAC will be † equal to the Segment EDF. But the whole Circle ABC is equal to the whole Circle DEF. Therefore the remaining Circumference BKC

* 4. 1.

† *Def.* 11.

‡ 24 *of this.*

† *Def.* 11.

ſhall be equal to the remaining Circumference E L F. Therefore *in equal Circles, equal Angles ſtand upon equal Circumferences, whether they be at their Centres, or at their Circumferences*; which was to be demonſtrated.

PROPOSITION XXVII.

THEOREM.

Angles that ſtand upon equal Circumferences in equal Circles, are equal to each other, whether they be at their Centres or Circumferences.

LET the Angles BGC, EHF, at the Centres of the equal Circles A B C, D E F, and the Angles BAC, EDF, at their Circumferences, ſtand upon the equal Circumferences B C, E F. I ſay, the Angle BGC is equal to the Angle EHF, and the Angle B A C to the Angle E D F.

For if the Angle BGC be equal to the Angle EHF, it is manifeſt that the Angle BAC is alſo equal to the Angle E D F: But if not, let one of them be the greater, as B G C, and make * the Angle BGK, at the Point G, with the Line BG, equal to the Angle E H F. But equal Angles ſtand † upon equal Circumferences, when they are at the Centres. Wherefore the Circumference BK is equal to the Circumference E F. But the Circumference E F is equal to the Circumference BC. Therefore BK is equal to BC, a leſs to a greater; which is abſurd. Wherefore the Angle BGC is not unequal to the Angle EHF; and ſo it muſt be equal to it. But the Angle at A is one half of the Angle BGC; and the Angle at D one half of the Angle EHF. Therefore the Angle at A is equal to the Angle at D. Wherefore *Angles that ſtand upon equal Circumferences in equal Circles, are equal to each other, whether they be at their Centres or Circumferences*; which was to be demonſtrated.

* 23. 1.

† 26 *of this.*

PROPOSITION XXVIII.

THEOREM.

In equal Circles, equal Right Lines cut off equal Parts of the Circumferences; the greater equal to the greater, and the leſſer equal to the leſſer.

LET ABC, DEF, be equal Circles, in which are the equal Right Lines BC, EF, which cut off the greater Circumferences BAC, EDF, and the leſſer Circumferences BGC, EHF. I ſay, the greater Circumference BAC is equal to the greater Circumference EDF, and the leſſer Circumference BGC to the leſſer Circumference EHF.

For, aſſume the Centres K and L of the Circles, and join BK, KC, EL, LF.

Becauſe the Circles are equal, the Lines drawn from their Centres are * alſo equal. Therefore the * *Def.* 1.
two Sides BK, KC, are equal to the two Sides EL, LF; and the Baſe BC is equal to the Baſe EF. Therefore the Angle BKC is † equal to the Angle † 8. 1.
ELF. But equal Angles ſtand ‡ upon equal Circum- ‡ 26 *of this.*
ferences, when they are at the Centres. Wherefore the Circumference BGC is equal to the Circumference EHF, and the whole Circle ABC equal to the whole Circle DEF; and ſo the remaining Circumference BAC ſhall be equal to the remaining Circumference EDF. Therefore in equal Circles, equal Right Lines cut off equal Parts of the Circumferences; *which was to be demonſtrated.*

PROPOSITION XXIX.

THEOREM.

In equal Circles, equal Right Lines ſubtend equal Circumferences.

LET there be two equal Circles, ABC, DEF; and let the equal Circumferences BGC, EHF, be aſſumed in them, and BC, EF, joined. I ſay, the Right Line BC is equal to the Right Line EF.

* 1 *of this.* For, find * the Centres of the Circles K, L; and join BK, KC, EL, LF.

Then, becauſe the Circumference BGC is equal to the Circumference EHF; the Angle BKC ſhall be † equal to the Angle ELF. And becauſe the Circles ABC, DEF, are equal, the Lines drawn from their Centres ſhall be ‡ equal. Therefore the two Sides BK, KC, are equal to the two Sides EL, LF; and they contain equal Angles: Wherefore the Baſe BC is ‡ equal to the Baſe EF. And ſo, in equal Circles, equal Circumferences ſubtend equal Right Lines; *which was to be demonſtrated.*

† 27 *of this.*
‡ *Def.* 1.
‡ 4. 1.

PROPOSITION XXX.

PROBLEM.

To cut a given Circumference into two equal Parts.

LET the given Circumference be ADB. It is required to cut the ſame into two equal Parts.

* 10. 1. Join AB, which biſect * in C; and let the Right Line CD be drawn from the Point C at Right Angles to AB; and join AD, DB.

Now, becauſe AC is equal to CB, and CD is common, the two Sides AC, CD, are equal to the two Sides BC, CD; but the Angle ACD is equal to the Angle BCD; for each of them is a Right Angle: Therefore the Baſe AD is † equal to the Baſe BD. But equal Right Lines cut ‡ off equal Circumferences. Wherefore the Circumference AD ſhall be equal to the Circumference BD. Therefore a given Circumference is cut into two equal Parts; *which was to be done.*

* 4. 1.
‡ 28 *of this.*

PRO-

PROPOSITION XXXI.

THEOREM.

In a Circle, the Angle that is in a Semicircle, is a Right Angle; but the Angle in a greater Segment, is leſs than a Right Angle; and the Angle in a leſſer Segment, greater than a Right Angle: Moreover, the Angle of a greater Segment is greater than a Right Angle; and the Angle of a leſſer Segment is leſs than a Right [redacted]. *Angle.*

LET there be a Circle ABCD, whoſe Diameter is BC, and Centre E; and join BA, AC, AD, DC. I ſay, the Angle which is in the Semicircle BAC is a Right Angle; that which is in the Segment ABC being greater than a Semicircle, *viz.* the Angle ABC, is leſs than a Right Angle; and that which is in the Segment ADC, being leſs than a Semicircle, that is, the Angle ADC, is greater than a Right Angle.

For, join AE, and produce BA to F.

Then, becauſe BE is equal to EA, the Angle EAB
ſhall be * equal to the Angle EBA. And becauſe * 5. 1.
AE is equal to EC, the Angle ACE will be * equal
to the Angle CAE. Therefore the whole Angle
BAC is equal to the two Angles ABC, ACB;
but the Angle FAC, being without the Triangle
ABC, is † equal to the two Angles ABC, ACB: † 32. 1.
Therefore the Angle BAC is equal to the Angle
FAC; and ſo each of them is ‡ a Right Angle. ‡ *Def.* 10. 1.
Wherefore the Angle BAC in a Semicircle is a Right
Angle. And becauſe the two Angles ABC, BAC,
of the Triangle ABC *, are leſs than two Right * 17. 1.
Angles, and BAC is a Right Angle; then ABC is
leſs than a Right Angle, and is in the Segment ABC
greater than a Semicircle.

And ſince ABCD is a quadrilateral Figure in a
Circle, and the oppoſite Angles of any quadrilateral
Figure deſcribed in a Circle, are † equal to two Right † 22 *of this.*
Angles; the Angles ABC, ADC, are equal to two
Right Angles, and the Angle ABC is leſs than a
Right Angle: Therefore the remaining Angle ADC
will

will be greater than a Right Angle, and is in the Segment A D C, which is leſs than a Semicircle.

I ſay, moreover, the Angle of the greater Segment contained under the Circumference A B C, and the Right Line A C, is greater than a Right Angle; and the Angle of the leſſer Segment, contained under the Circumference A D C, and the Right Line A C, is leſs than a Right Angle. This manifeſtly appears; for, becauſe the Angle contained under the Right Lines B A, A C, is a Right Angle, the Angle contained under the Circumference A B C, and the Right Line A C, will be greater than a Right Angle. Again, becauſe the Angle contained under the Right Line C A, A F, is a Right Angle, therefore the Angle which is contained under the Right Line A C, and the Circumference A D C, is leſs than a Right Angle. Therefore, *in a Circle, the Angle that is in a Semicircle, is a Right Angle; but the Angle in a greater Segment is leſs than a Right Angle; and the Angle in a leſſer Segment greater than a Right Angle: Moreover, the Angle of a greater Segment is greater than a Right Angle; and the Angle of a leſſer Segment is leſs than a Right Angle*; which was to be demonſtrated.

PROPOSITION XXXII.

THEOREM.

If any Right Line touches a Circle, and a Right Line be drawn from the Point of Contact cutting the Circle; the Angles it makes with the Tangent Line, will be equal to thoſe which are made in the alternate Segments of the Circle.

LET any Right Line E F touch the Circle A B C D in the Point B, and let the Right Line B D be any how drawn from the Point B cutting the Circle. I ſay, the Angles which B D makes with the Tangent Line E F, are equal to thoſe in the alternate Segments of the Circle; that is, the Angle F B D is equal to an Angle made in the Segment D A B, *viz.* to the Angle D A B; and the Angle D B E equal to the Angle D C B, made in the Segment D C B. For,

Draw * B A from the Point B at Right Angles to * 11. 1.
E F; and take any Point C in the Circumference
B D, and join A D, D C, C B.

Then, becauſe the Right Line E F touches the Circle
A B C D in the Point B; and the Right Line B A
is drawn from the Point of Contact B at Right Angles
to the Tangent Line; the Centre of the Circle
A B C D will † be in the Right Line B A; and ſo † 19. 3.
B A is a Diameter of the Circle, and the Angle A D B,
in a Semicircle, is ‡ a Right Angle. Therefore the ‡ 31 *of this.*
other Angles B A D, A B D, are * equal to one Right * 32. 1.
Angle. But the Angle A B F is alſo a Right Angle:
Therefore the Angle A B F is equal to the Angles
B A D, A B D; and if A B D, which is common, be
taken away, then the Angle D B F remaining, will
be equal to that which is in the alternate Segment of
the Circle, *viz.* equal to the Angle B A D. And becauſe
A B C D is a quadrilateral Figure in a Circle,
and the oppoſite Angles thereof are † equal to two † 22 *of this.*
Right Angles; the Angles D B F, D B E, will be
equal to the Angles B A D, B C D. But B A D has
been proved to be equal to D B F; therefore the Angle
D B E is equal to the Angle made in D C B, the
alternate Segment of the Circle, *viz.* equal to the
Angle D C B. Therefore, *if any Right Line touches
a Circle, and a Right Line be drawn from the Point
of Contact cutting the Circle; the Angles it makes
with the Tangent Line, will be equal to thoſe which
are made in the alternate Segments of the Circle;*
which was to be demonſtrated.

PROPOSITION XXXIII.

PROBLEM.

To deſcribe, upon a given Right Line, a Segment of a Circle, which ſhall contain an Angle equal to a given Right-lined Angle.

LET the given Right Line be A B, and C the given Right-lined Angle. It is required to deſcribe the Segment of a Circle upon the given Right Line A B, containing an Angle equal to the Angle C.

At the Point A, with the Right Line AB, make ‡ the Angle BAD equal to the Angle C, and draw * AE from the Point A, at Right Angles to AD. Likewiſe biſect † AB in F, and let FG be drawn from the Point F, at Right Angles, to AB, and join GB.

‡ 23. 1.
* 11. 1.
† 10. 1.

Then, becauſe AF is equal to FB, and FG is common, the two Sides AF, FG, are equal to the two Sides BF, FG; and the Angle AFG is equal to the Angle BFG. Therefore the Baſe AG is ‡ equal to the Baſe GB. And ſo, if a Circle be deſcribed about the Centre G, with the Diſtance AG, this ſhall paſs through the Point B. Deſcribe the Circle, which let be ABE, and join EB. Now, becauſe AD is drawn from the Point A, the Extremity of the Diameter AE, at Right Angles to AE, the ſaid AD will * touch the Circle. And ſince the Right Line AD touches the Circle ABE, and the Right Line AB is drawn in the Circle from the Point of Contact A, the Angle DAB is † equal to the Angle made in the alternate Segment, *viz.* equal to the Angle AEB. But the Angle DAB is equal to the Angle C. Therefore the Angle C will be equal to the Angle AEB. Wherefore the Segment of a Circle AEB is deſcribed upon the given Right Line AB, containing an Angle AEB, equal to a given Angle C; *which was to be done.*

‡ 4. 1.
* Cor. 16. of this.
† 32 of this.

PROPOSITION XXXIV.

THEOREM.

To cut off a Segment from a given Circle, that ſhall contain an Angle equal to a given Right-lin'd Angle.

LET the given Circle be ABC, and the Right-lined Angle given D. It is required to cut off a Segment from the Circle ABC, containing an Angle equal to the Angle D.

Draw ‡ the Right Line EF, touching the Circle in the Point B, and make * the Angle FBC at the Point B equal to the Angle D.

‡ 17 of this.
* 23. 1.

Then, becauſe the Right Line EF touches the Circle ABC in the Point B, and BC is drawn from the Point of Contact B; the Angle F B C will be *equal * 32 *of this*, to that in the alternate Segment of the Circle; but the Angle F B C is equal to the Angle D. Therefore the Angle in the Segment B A C will be equal to the Angle D. Therefore the Segment B A C is cut off from the given Circle A B C, containing an Angle equal to the given Right-lined Angle D; *which was to be done.*

PROPOSITION XXXV.

THEOREM.

If two Right Lines in a Circle mutually cut each other, the Rectangle contained under the Segments of the one, is equal to the Rectangle under the Segments of the other.

IN the Circle A B C D, let two Right Lines mutually cut each other in the Point E. I ſay, the Rectangle contained under A E, and E C, is equal to the Rectangle contained under D E, E B.

If A C and D B paſs through the Centre, ſo that E be the Centre of the Circle A B C D; it is manifeſt, ſince A E, E C, D E, E B, are equal; that the Rectangle under A E, E C, is equal to the Rectangle under D E, E B.

But if A C, D B, do not paſs through the Centre, aſſume the Centre of the Circle F; from which draw F G, F H, perpendicular to the Right Lines A C, D B; and join F B, F C, F E.

Then, becauſe the Right Line G F, drawn through the Centre, cuts the Right Line A C, not drawn thro' the Centre, at Right Angles, it will alſo biſect * the * 4 *of this*. ſame. Wherefore A G is equal to G C: And becauſe the Right Line A C is cut into two equal Parts in the Point G, and into two unequal Parts in E, the Rectangle under A E, E C, together with the Square of E G, is † equal to the Square of G C. And if † 5. 2. the common Square of G F be added, then the Rectangle under A E, E C, together with the Squares of E G, G F, is equal to the Squares of C G, G F. But the Square of F E is † equal to the Squares of E G, G F, and the Square of F C equal ‡ to the Squares ‡ 47. 1.

of CG, GF. Therefore the Rectangle under AE, EC, together with the Square of FE, is equal to the Square of FC, but CF is equal to FB. Therefore the Rectangle under AE, EC, together with the Square of EF, is equal to the Square of FB. For the ſame Reaſon, the Rectangle under DE, EB, together with the Square of FE, is equal to the Square of FB. But it has been proved, that the Rectangle under AE, EC, together with the Square of FE, is alſo equal to the Square of FB: Therefore the Rectangle under AE, EC, together with the Square of FE, is equal to the Rectangle under DE, EB, together with the Square of FE. And if the common Square of FE be taken away, then there will remain the Rectangle under AE, EC, equal to the Rectangle under DE, EB. Wherefore, *if two Right Lines in a Circle mutually cut each other, the Rectangle contained under the Segments of the one, is equal to the Rectangle under the Segments of the other*; which was to be demonſtrated.

PROPOSITION XXXVI.

THEOREM.

If ſome Point be taken without a Circle, and from that Point two Right Lines fall to the Circle, one of which cuts the Circle, and the other touches it; the Rectangle contained under the whole Secant Line, and its Part between the Convexity of the Circle and the aſſumed Point, will be equal to the Square of the Tangent Line.

LET any Point D be aſſumed without the Circle ABC, and let two Right Lines DCA, DB, fall from the ſaid Point to the Circle; whereof DCA cuts the Circle, and DB touches it. I ſay, the Rectangle under AD, DC, is equal to the Square of DB.

Now, DCA either paſſes thro' the Centre, or not. In the firſt Place, let it paſs thro' the Centre of the Circle ABC, which let be E, and join EB. Then the Angle EBD is * a Right Angle. And ſo, ſince the Right Line AC is biſected in E, and CD is added thereto, the Rectangle under AD, DC, together with

* 18 *of this.*

with the Square of E C, ſhall * be equal to the Square of E D. But E C is equal to E B; wherefore the Rectangle under A D, D C, together with the Square of E B, is equal to the Square of E D. But the Square of E D is † equal to the Square of E B, and B D. For the Angle E B D is a Right Angle: Therefore the Rectangle under A D, D C, together with the Square of E B, is equal to the Squares of E B and B D; and if the common Square of E B be taken away, the Rectangle under A D, D C, remaining, will be equal to the Square of the Tangent Line B D. * 6. 2. † 47. 1.

Now, let D C A not paſs through the Centre of the Circle A B C; and find ‡ the Centre E thereof, and draw E F perpendicular to A C, and join E B, E C, E D. Therefore E F D is a Right Angle. And becauſe a Right Line E F, drawn through the Centre, cuts a Right Line A C at Right Angles, not drawn through the Centre, it will * biſect the ſame at Right Angles; and ſo A F is equal to F C. Again, ſince the Right Line A C is biſected in F, and C D is added thereto, the Rectangle under A D, D C, together with the Square of F C, will be * equal to the Square of F D. And if the common Square of E F be added, then the Rectangle under A D, D C, together with the Squares of F C and F E, is equal to the Squares of D F and F E. But the Square of D E is equal to the Squares of D F and F E; for the Angle E F D is a Right one: And the Square of C E is † equal to the Square of C F and F E. Therefore the Rectangle under A D, D C, together with the Square of C E, is equal to the Square of E D; but C E is equal to E B. Wherefore the Rectangle under A D, D C, together with the Square of E B, is equal to the Square of E D. But the Squares of E B and B D are † equal to the Square of E D; ſince the Angle E B D is a Right one. Wherefore the Rectangle under A D and D C, together with the Square of E B, is equal to the Squares of E B and B D. And if the common Square of E B be taken away, the Rectangle under A D and D C, remaining, will be equal to the Square of D B. Therefore, *if any Point be taken without a Circle, and from that Point two Right Lines fall to the Circle, one of which cuts the Circle, and the other touches it; the Rectangle contained under the whole Secant Line, and its* ‡ 1 *of this.* * 3 *of this.*

its Part between the Convexity of the Circle and the aſſumed Point, will be equal to the Square of the Tangent Line; which was to be demonſtrated.

PROPOSITION XXXVII.

THEOREM.

If ſome Point be taken without a Circle, and two Right Lines be drawn from it to the Circle, ſo that one cuts it, and the other falls upon it; and if the Rectangle under the whole Secant Line, and the Part thereof, without the Circle, be equal to the Square of the Line falling upon the Circle, then this laſt Line will touch the Circle.

LET ſome Point D be aſſumed without the Circle ABC, and from it draw two Right Lines DCA, DB, to the Circle, in ſuch manner that DCA cuts the Circle; and DB falls upon it: And let the Rectangle under AD, DC, be equal to the Square of DB. I ſay, the Right Line DB touches the Circle.

* 17 *of this.* For, let the Right Line DE be drawn * touching the Circle ABC, and find F the Centre of the Circle, and join EF, FB, FD.

† 18 *of this.* Then, the Angle FED is † a Right Angle. And becauſe DE touches the Circle ABC, and DCA cuts it, the Rectangle under AD, and DC, will be equal to the Square of DE. But the Rectangle under AD and DC is ‡ equal to the Square of DB. ‡ *By Hyp.* Wherefore the Square of DE ſhall be equal to the Square of DB. And ſo the Line DE will be equal to the Line DB. But EF is equal to FB: Therefore the two Sides DE, EF, are equal to the two Sides DB, BF; and the Baſe FD is common. Wherefore the Angle DEF is equal to the Angle DBF; but DEF is a Right Angle; wherefore DBF is alſo a Right Angle, and FB produced is a Diameter. But a Right Line drawn at Right Angles, on the End of the Diameter of a Circle, touches the Circle; therefore BD neceſſarily touches the Circle. We prove this in the ſame manner, if the Centre of the Circle be in the Right Line CA. *If* therefore *any Point be aſſumed without a Circle, and two Right Lines*

Lines be drawn from it to the Circle, ſo that one cuts it, and the other falls upon it; and if the Rectangle under the whole Secant Line, and the Part thereof, without the Circle, be equal to the Square of the Line falling upon the Circle; then this laſt Line will touch the Circle; which was to be demonſtrated.

Coroll. Hence, if from any Point without a Circle, ſeveral Right Lines AB, AC, are drawn cutting the Circle; the Rectangles comprehended under the whole Lines AB, AC, and their external Parts AE, AF, are equal between themſelves. For, if the Tangent AD be drawn, the Rectangle under BA and AE, is equal to the Square of AD; and the Rectangle under CA and AF, is equal to the ſame Square of AD: Therefore the Rectangles ſhall be equal.

The END *of the* THIRD BOOK.

EUCLID's

ELEMENTS.

BOOK IV.

DEFINITIONS.

I. *A Right-lined Figure is ſaid to be inſcribed in a Right-lined Figure, when every one of the Angles of the inſcribed Figure, touches every one of the Sides of the Figure, wherein it is deſcribed.*

II. *In like manner a Figure is ſaid to be deſcribed about a Figure, when every one of the Sides of the Figure, circumſcribed, touches every one of the Angles of the Figure, about which it is circumſcribed.*

III. *A Right-lined Figure is ſaid to be inſcribed in a Circle, when every one of the Angles of that Figure which is inſcribed, touches the Circumference of the Circle.*

IV. *A Right-lined Figure is ſaid to be deſcribed about a Circle, when every one of the Sides of the circumſcribed Figure, touches the Circumference of the Circle.*

V. *So*

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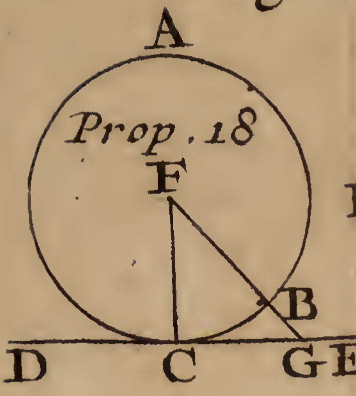
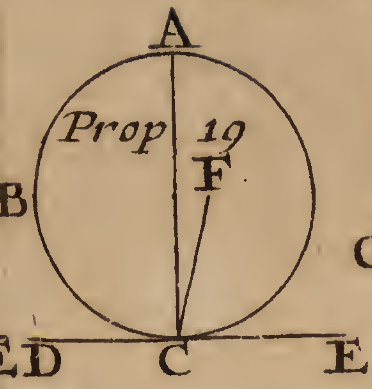
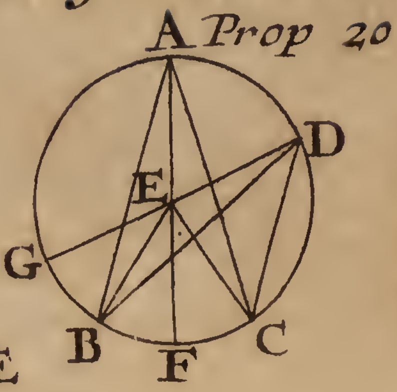
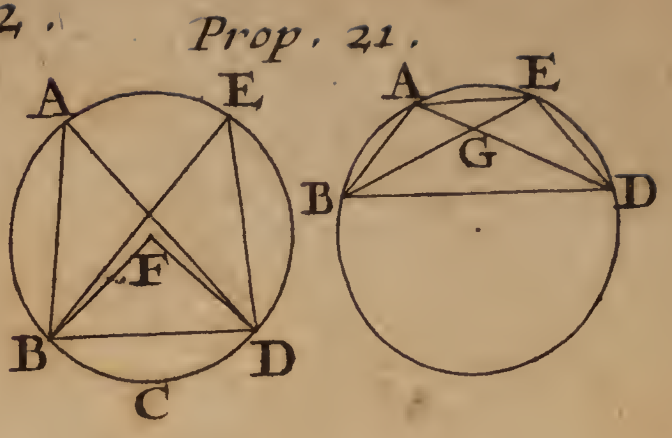
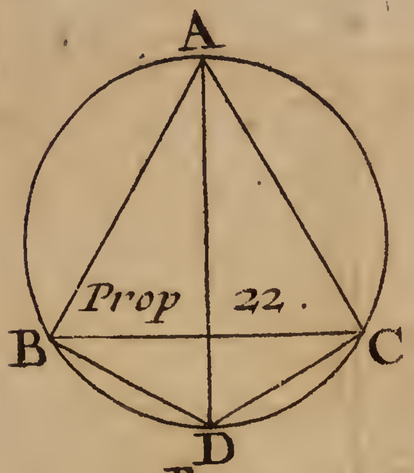
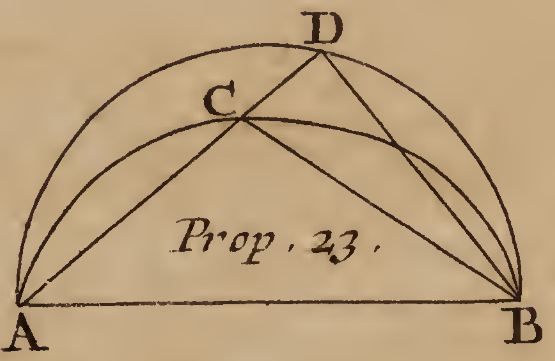


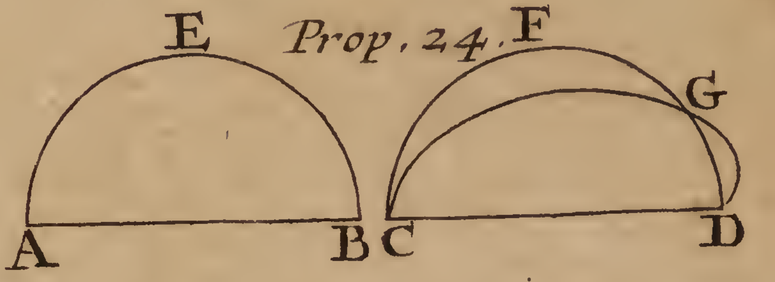


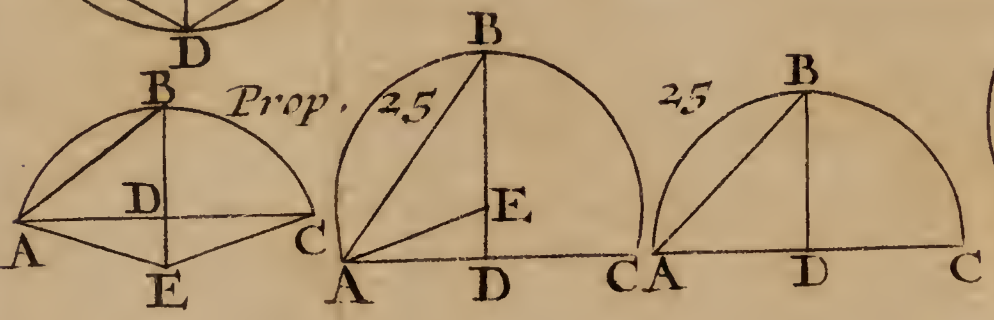


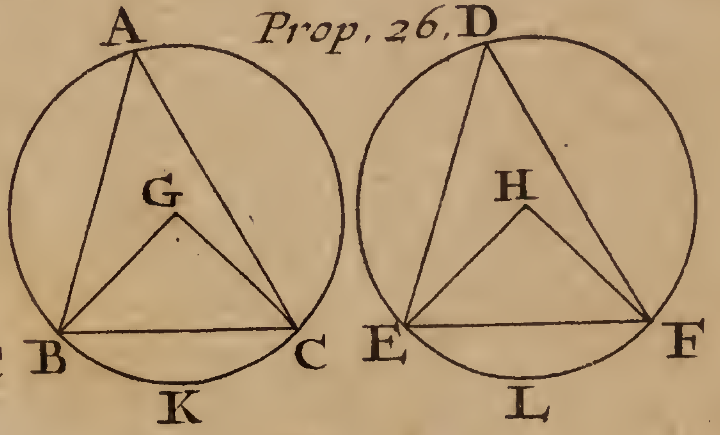


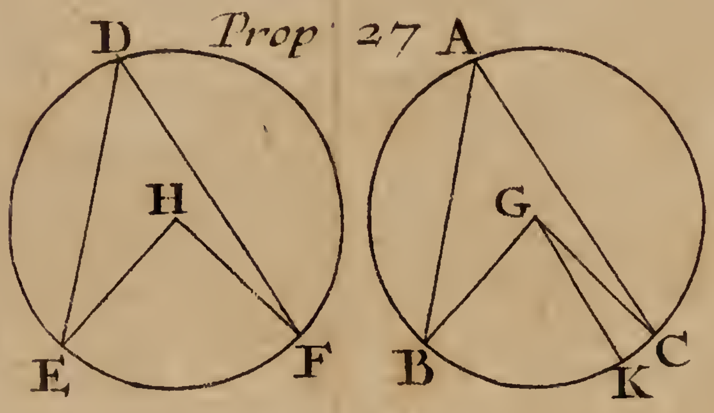


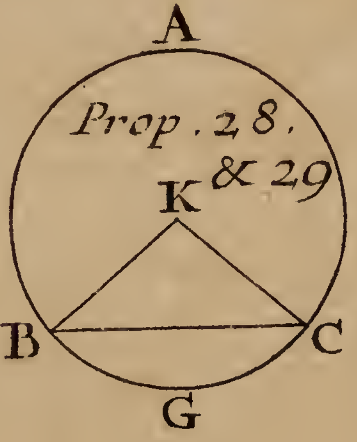


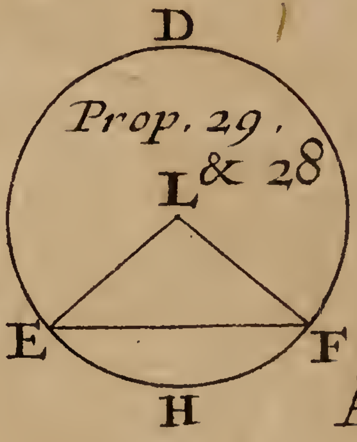


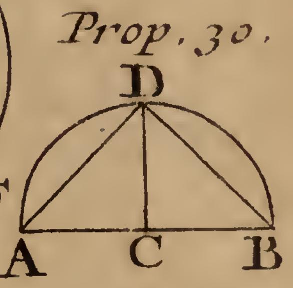


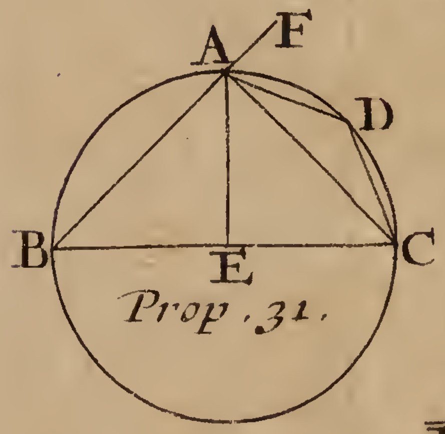


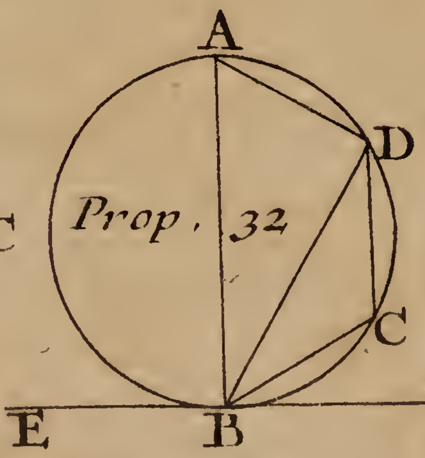


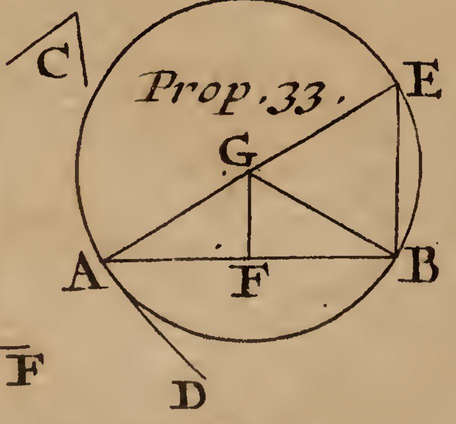


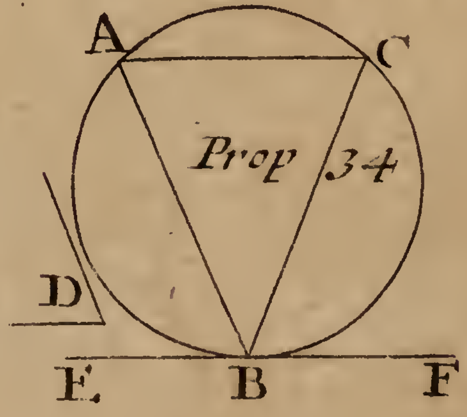


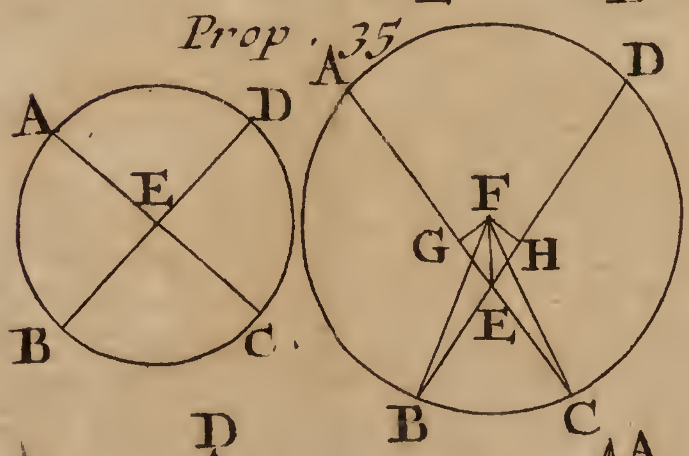


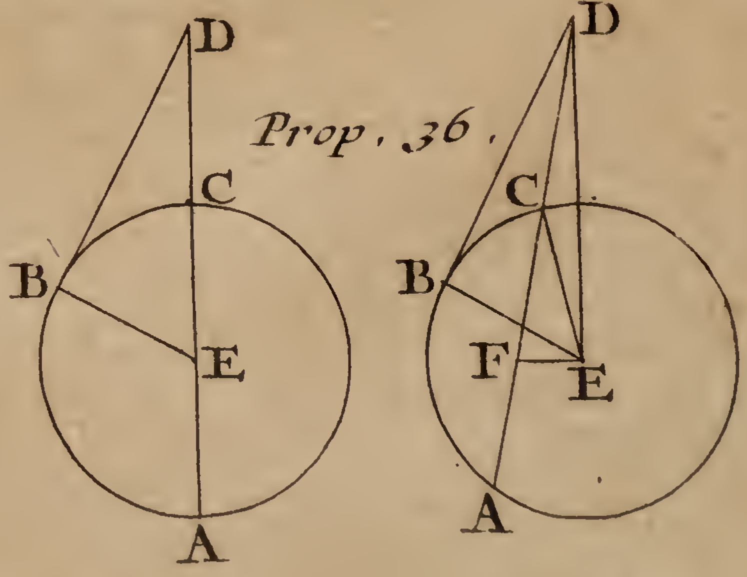


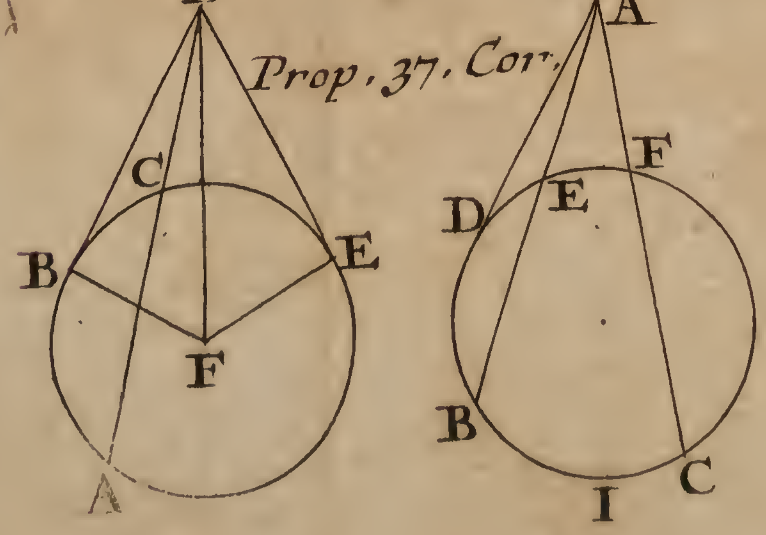












V. *So likewiſe a Circle is ſaid to be inſcribed in a Right-lined Figure, when the Cirumference of the Circle touches all the Sides of the Figure in which it is inſcribed.*

VI. *A Circle is ſaid to be deſcribed about a Figure, when the Circumference of the Circle touches all the Angles of the Figure which it circumſcribes.*

VII. *A Right Line is ſaid to be applied in a Circle, when its Extremes are in the Circumference of the Circle.*

PROPOSITION I.

PROBLEM.

To apply a Right Line in a given Circle, equal to a given Right Line, whoſe Length does not exceed the Diameter of the Circle.

LET the Circle given be ABC, and the given Right Line not greater than the Diameter be D. It is required to apply a Right Line in the Circle ABC, equal to the Right Line D.

Draw BC the Diameter of the Circle; then, if BC be equal to D, what was required is done: For in the Circle ABC there is applied the Right Line BC, equal to the Right Line D: But if not, the Diameter BC is greater than D, and put * CE equal to D; and *3. 1. about the Centre C, with the Diſtance CE, let the Circle AEF be deſcribed; and join CA.

Then, becauſe the Point C is the Centre of the Circle AEF, CA will be equal to CE; but D is equal to CE. Wherefore AC is equal to D. And ſo in the Circle ABC, there is applied a Right Line AC, equal to the given Right Line D, not greater than the Diameter; *which was to be done.*

PROPOSITION II.

PROBLEM.

In a given Circle, to deſcribe a Triangle equiangular to a given Triangle.

LET ABC be a Circle given, and DEF a given
Triangle. It is required to deſcribe a Triangle
in the Circle ABC, equiangular to the Triangle DEF.
* 17. 3. Draw the Right Line GAH touching * the Circle
ABC in the Point A, and with the Right Angle AH
† 23. 1. at the Point A make † an Angle HAC, equal to the
Angle DEF. Likewiſe at the ſame Point A, with
the Line AG, make the Angle GAB equal to the
Angle DFE, and join BC.

Then, becauſe the Right Line HAG touches the
Circle ABC, and AC is drawn from the Point of
Contact in the Circle; the Angle HAC ſhall be
‡ 32. 3. ‡ equal to ABC, the Angle in the alternate Segment
of the Circle. But the Angle HAC is equal to the
Angle DEF; therefore alſo the Angle ABC is equal
to the Angle DEF: For the ſame Reaſon, the Angle
ACB is likewiſe equal to the Angle DFE. Where-
† *Cor.* 2. fore the other Angle BAC ſhall be † equal to the
32. 1. other Angle EDF. And conſequently, the Triangle
ABC is equiangular to the Triangle DEF, and is
deſcribed in the Circle ABC; *which was to be done.*

PROPOSITION III.

PROBLEM.

About a given Circle to deſcribe a Triangle, equiangular to a Triangle given.

LET ABC be the given Circle, and DEF the given Triangle. It is required to deſcribe a Triangle about the Circle ABC, equiangular to the Triangle DEF.

Produce the Side EF both ways to the Points G and H, and find the Centre of the Circle K, and anyhow draw the Line KB. Then at the Point K, with

KB make * the Angle BKA equal to the Angle * 23. 1. DEG; and the Angle BKC at the ſame Point K on the other Side the Line KB, equal to the Angle DFH; and thro' the Points A, B, C, let the Right Lines LAM, MBN, NCL, be drawn touching the Circle ABC.

Then, becauſe the Lines LM, MN, NL, touch the Circle ABC in the Points A, B, C, and the Lines KA, KB, KC, are drawn from the Centre K to the Points A, B, C; the Angles at the Points A, B, C, will be † Right Angles. And becauſe the four An- † 18. 3. gles of the quadrilateral Figure AMBK are equal to four Right Angles, (for it may be divided into two Triangles) and the Angles KAM, KBM, are each Right Angles; therefore the other Angles AKB, AMB, are equal to two Right Angles. But DEG, DEF, are equal to two Right Angles; therefore the Angles AKB, AMB, are equal to the Angles DEG, DEF, whereof AKB is equal to DEG. Wherefore the other Angle AMB is equal to the other Angle DEF. In like manner we demonſtrate, that the Angle LNB is equal to the Angle DFE. Therefore the other Angle MLN is ‡ equal to the other Angle ‡ *Cor.* 2, 32. 1. EDF. Wherefore the Triangle LNM is equiangular to the Triangle DEF, and is deſcribed about the Circle ABC; *which was to be done.*

PROPOSITION IV.

PROBLEM.

To inſcribe a Circle in a given Triangle.

LET ABC be a Triangle given. It is required to inſcribe a Circle in the ſame.

Cut * the Angles ABC, BCA, into two equal * 9. 1. Parts by the Right Lines BD, DC, meeting each other in the Point D. And from this Point draw DE, DF, DG, † perpendicular to the Sides AB, † 12. 1. BC, AC.

Now, becauſe the Angle EBD is equal to the Angle FBD, and the Right Angle BED is equal to the Right Angle BFD; then the two Triangles EBD, DBF, have two Angles of the one, equal to two

Angles of the other, and one Side DB common to both, *viz.* that which ſubtends the equal Angles; therefore ‡ 26. 1. the other Sides of the one Triangle ſhall be ‡ equal to the other Sides of the other; and ſo D E ſhall be equal to D F. And for the ſame Reaſon, D G is equal to D F: Therefore D E is alſo equal to D G. And ſo the three Right Lines D E, D F, D G, are equal between themſelves. Wherefore a Circle deſcribed about the Centre D, with either of the Diſtances D E, D F, D G, will alſo paſs thro' the other Points. And the Sides A B, B C, A C, will touch it; becauſe the Angles at E, F, and G, are Right Angles. For if it ſhould cut them, a Right Line drawn on the Extremity of the Diameter of a Circle at Right Angles, will fall within the Circle; * 16. 3. which is * abſurd. Therefore a Circle deſcribed about the Centre D, with either of the Diſtances D E, D F, D G, will not cut the Sides A B, B C, C A; wherefore it will touch them, and will be a Circle deſcribed in the Triangle A B C. Therefore the Circle E F G is deſcribed in the given Triangle A B C; *which was to be done.*

PROPOSITION V.

PROBLEM.

To deſcribe a Circle about a given Triangle.

LET A B C be a given Triangle. It is required to deſcribe a Circle about the ſame.

* 10. 1. Biſect * the Sides A B, A C, in the Points D, E; † 11. 1. from which Points let D F, E F, be drawn † at Right Angles to A B, A C, which will meet either within the Triangle A B C, or in the Side B C, or without the Triangle.

Firſt, let them meet in the Point F within the Triangle, and join B F, F C, F A. Then, becauſe A D is equal to D B, and D F is common, and at Right Angles to A B; ‡ 4. 1. the Baſe A F will be ‡ equal to the Baſe F B. And after the ſame manner we prove, that the Baſe C F is equal to the Baſe F A. Therefore alſo is B F equal to C F: And ſo the three Right Lines F A, F B, F C, are equal to each other. Wherefore a Circle deſcribed about the Centre F, with either of the

Diſtances F A, F B, F C, will paſs alſo thro' the other Points, and will be a Circle deſcribed about the Triangle A B C. Therefore deſcribe the Circle A B C.

Secondly, let D F, E F, meet each other in the Point F, in the Side B C, as in the ſecond Figure, and join A F. Then we prove, as before, that the Point F is the Centre of a Circle deſcribed about the Triangle A B C.

Laſtly, let the Right Lines D F, E F, meet one another again in the Point F, without the Triangle, as in the third Figure; and join A F, F B, F C And becauſe A D is equal to D B, and D F is common, and at Right Angles, the Baſe A F ſhall be equal to the Baſe B F. So likewiſe we prove, that C F is alſo equal to A F. Wherefore B F is equal to C F. And ſo again, if a Circle be deſcribed on the Centre F, with either of the Diſtances F A, F B, F C, it will paſs through the other Points, and will be deſcribed about the Triangle A B C; *which was to be done.*

Coroll. If a Triangle be Right-angled, the Centre of the Circle falls in the Side oppoſite to the Right Angle; if acute-angled, it falls within the Triangle; and if obtuſe-angled, it falls without the Triangle.

PROPOSITION VI.

PROBLEM.

To inſcribe a Square in a given Circle.

LET ABCD be a Circle given. It is required to inſcribe a Square within the ſame.

Draw A C, B D, two Diameters of the Circle cutting one another at Right Angles, and join A B, B C, C D, D A.

Then, becauſe B E is equal to E D, (for E is the Centre) and E A is common, and at Right Angles to B D, the Baſe B A ſhall be * equal to the Baſe A D; * 4. 1.
and for the ſame Reaſon B C, C D, as alſo B A, A D, are all equal to each other. Therefore the quadrilateral Figure A B C D is equilateral. I ſay, it is alſo rectangular. For, becauſe the Right Line D B is a Diameter of the Circle A B C D, B A D will be a Se-

* 31. 3. micircle. Wherefore the Angle BAD is * a Right
Angle. And for the ſame Reaſon every one of the Angles ABC, BCD, CDA, is a Right Angle. Therefore ABCD is a rectangular quadrilateral Figure: But it has alſo been proved to be equilateral. Wherefore it ſhall neceſſarily be a Square, and is deſcribed in the Circle ABCD; *which was to be done.*

PROPOSITION VII.

PROBLEM.

To deſcribe a Square about a given Circle.

LET ABCD be a Circle given. It is required to deſcribe a Square about the ſame.

Draw AC, BD, two Diameters of the Circle
cutting each other at Right Angles; and through the
* 17. 3. Points A, B, C, D, draw * FG, GH, HK, KF,
Tangents to the Circle ABCD.

Then, becauſe FG touches the Circle ABCD, and
EA is drawn from the Centre E to the Point of Con-
† 18. 3. tact A, the Angles at A will be † Right Angles. For
the ſame Reaſon, the Angles at the Points B, C, D,
are Right Angles. And ſince the Angle AEB is a
‡ 28. 1. Right Angle, as alſo EBG, GH ſhall be ‡ parallel
to AC; and for the ſame Reaſon, AC to KF. In
this manner we prove likewiſe, that GF and HK
are parallel to BED; and ſo GF is parallel to HK.
Therefore GK, GC, AK, FB, BK, are Parallelo-
* 34. 1. grams; and ſo GF is * equal to HK, and GH to
FK. And ſince AC is equal to BD, and AC * equal
to either GH, or FK; and BD equal to either GF,
or HK; GH, or FK, is equal to GF, or HK. There-
fore FGHK is an equilateral quadrilateral Figure:
I ſay it is alſo equiangular. For, becauſe GBEA is
a Parallelogram, and AEB is a Right Angle, then
AGB ſhall be alſo a Right Angle. In like manner
we demonſtrate, that the Angles at the Points H, K, F,
are Right Angles. Therefore the quadrilateral Figure
FGHK is rectangular; but it has been proved to be
equilateral likewiſe. Wherefore it muſt neceſſarily be
a Square, and is deſcribed about the Circle ABCD;
which was to be done.

PRO-

PROPOSITION VIII.

PROBLEM.

To deſcribe a Circle in a given Square.

LET the given Square be ABCD. It is required to deſcribe a Circle within the ſame.

Biſect * the Sides AB, AD, in the Points F, E; * 10. 1.
and draw † EH thro' E, parallel to AB, or DC; and † 31. 1.
FK thro' F, parallel † to BC, or AD. Then AK,
KB, AH, HD, AG, GC, BG, GD, are all Parallelo-
grams, and their oppoſite Sides are ‡ equal. And be- ‡ 34. 1.
cauſe DA is equal to AB, and AE is half of AD,
and AF half of AB, AE ſhall be equal to AF; but
the oppoſite Sides are alſo equal. Therefore FG is
equal to GE. In like manner we demonſtrate, that
GH, or GK, is equal to either FG, or GE. There-
fore GE, GF, GH, GK, are equal to each other:
And ſo a Circle being deſcribed about the Centre G,
with either of the Diſtances GE, GF, GH, GK,
will alſo paſs thro' the other Points, and ſhall touch
the Sides AB, BC, CD, DA, becauſe the Angles at
E, F, H, K, are ‖ Right Angles. For if the Circle
ſhould cut the Sides of the Square, a Right Line,
drawn from the End of the Diameter of a Circle at
Right Angles, will fall within the Circle; which
is * abſurd. Wherefore a Circle deſcribed about the * 16. 3.
Centre G, with either of the Diſtances GE, GF,
GH, GK, will not cut AB, BC, CD, DA, the
Sides of the Square. Wherefore it ſhall neceſſarily
touch them, and will be deſcribed in the Square
ABCD; *which was to be done.*

PROPOSITION IX.

PROBLEM.

To deſcribe a Circle about a Square given.

LET ABCD be a Square given. It is required to circumſcribe a Circle about the ſame.

Join AC, BD, mutually cutting one another in the Point E.

And ſince D A is equal to A B, and A C is common, the two Sides D A, A C, are equal to the two Sides B A, A C; but the Baſe D C is equal to the Baſe B C. Therefore the Angle D A C will * be equal to the Angle B A C: And conſequently the Angle D A B is biſected by the Right Line A C. In the ſame manner we prove, that each of the Angles A B C, B C D, C D A, are biſected by the Right Lines A C, D B. * 8. 1.

Then, becauſe the Angle D A B is equal to the Angle A B C, and the Angle E A B is half of the Angle D A B, and the Angle E B A half of the Angle A B C; the Angle E A B ſhall be equal to the Angle E B A: And ſo the Side E A is † equal to the Side E B. In like manner we demonſtrate, that each of the Right Lines, E C, E D, is equal to each of the Right Lines E A, E B. Therefore the four Right Lines E A, E B, E C, E D, are equal between themſelves. Wherefore a Circle being deſcribed about the Centre E, with either of the Diſtances E A, E B, E C, E D, will alſo paſs thro' the other Points, and will be deſcribed about the Square A B C D; *which was to be done.* † 6. 1.

PROPOSITION X.

PROBLEM.

To make an Iſoſceles Triangle, having each of the Angles at the Baſe double to the other Angle.

CUT * any given Right Line A B in the Point C, ſo that the Rectangle contained under A B, B C, be equal to the Square of A C; then about the Centre A, with the Diſtance A B, let the Circle B D E be deſcribed; and † in the Circle B D E apply the Right Line B D equal to A C; which is not greater than the Diameter. This being done, join D A, D C, and deſcribe ‡ a Circle A C D about the Triangle A D C. * 11. 2. † 1 *of this.* ‡ 5 *of this.*

Then, becauſe the Rectangle A B C is equal to the Square of A C, and A C is equal to B D, the Rectangle under A B, B C, ſhall be equal to the Square of B D. And becauſe ſome Point, B, is taken without the Circle A C D, and from that Point there fall two Right Lines, B C A, B D, to the Circle, one of which

cuts the Circle, and the other falls on it; and ſince
the Rectangle under AB, BC, is equal to the Square
of BD, the Right Line BD ſhall * touch the Circle * 37. 3.
ACD. And ſince BD touches it, and DC is drawn
from the Point of Contact D, the Angle BDC is
equal to the Angle in the alternate Segment of the
Circle, *viz.* equal † to the Angle DAC. And ſince † 32. 3.
the Angle BDC is equal to the Angle DAC; if
CDA, which is common, be added, the whole Angle
BDA is equal to the two Angles CDA, DAC.
But the outward Angle BCD is ‡ equal to CDA, ‡ 32. 1.
DAC. Therefore BDA is equal to BCD. But
the Angle BDA * is equal to the Angle CBD, because * 5. 1.
the Side AD is equal to the Side AB. Wherefore
DBA ſhall be equal to BCD: And ſo the three
Angles BDA, DBA, BCD, are equal to each other.
And ſince the Angle DBC is equal to the Angle
BCD, the Side BD is † equal to the Side DC. But † 6. 1.
BD is put equal to CA: Therefore CA is equal to
CD. And ſo the Angle CDA is equal to the Angle
DAC. Therefore the Angles CDA, DAC, taken
together, are double to the Angle DAC. But the
Angle BCD is equal to the Angles CDA, DAC.
Therefore the Angle BCD is double to the Angle
DAC. But BCD is equal to BDA, or DBA.
Wherefore BDA, or DBA, is double to BDA.
Therefore the Iſoſceles Triangle ABD is made, having
one of the Angles at the Baſe, double to the other
Angle; *which was to be done.*

PROPOSITION XI.

PROBLEM.

To deſcribe an equilateral and equiangular Pentagon in a given Circle.

LET ABCDE be a Circle given. It is required to deſcribe an equilateral and equiangular Pentagon in the ſame.

Make an Iſoſceles Triangle FGH, having * each * 10 *of this.*
of the Angles at the Baſe GH, double to the other
Angle F; and deſcribe the Triangle ADC in the Circle
ABCDE, equiangular † to the Triangle FGH; ſo † 2 *of this.*
that

that the Angle CAD be equal to that at F, and ACD, CDA, each equal to the Angles G or H. Wherefore the Angles ACD, CDA, are each double to the Angle CAD. This being done, biſect * ACD, CDA, by the Right Lines CE, DB, and join AB, BC, DE, EA. * 9. 1.

Then, becauſe each of the Angles ACD, CDA, is double to CAD, and they are biſected by the Right Lines CE, DB; the five Angles DAC, ACE, ECD, CDB, BDA, are equal to each other. But equal Angles ſtand * upon equal Circumferences. Therefore the five Circumferences AB, BC, CD, DE, EA, are equal to each other. But equal Circumferences ſubtend † equal Right Lines. Therefore the five Right Lines AB, BC, CD, DE, EA, are equal to each other. Wherefore ABCDE is an equilateral Pentagon. I ſay, it is alſo equiangular; for becauſe the Circumference AB is equal to the Circumference DE, by adding the Circumference BCD, which is common, the whole Circumference ABCD is equal to the whole Circumference EDCB; but the Angle AED ſtands on the Circumference ABCD, and BAE on the Circumference EDCB: Therefore the Angle BAE is equal to the Angle AED. For the ſame Reaſon, each of the Angles ABC, BCD, CDE, is equal to BAE, or AED. Wherefore the Pentagon ABCDE is equiangular; but it has been proved to be alſo equilateral. And conſequently there is an equilateral and equiangular Pentagon inſcribed in a given Circle; *which was to be done.* * 26. 3. †

PROPOSITION XII.

PROBLEM.

To deſcribe an equilateral and equiangular Pentagon about a Circle given.

LET ABCDE be the given Circle. It is required to deſcribe an equilateral and equiangular Pentagon about the ſame.

Let A, B, C, D, E, be the angúlar Points of a Pentagon ſuppoſed to be inſcribed * in the Circle; ſo that the Circumferences AB, BC, CD, DE, EA, be equal; * By 11. of this.

equal; and let the Right Lines G H, H K, K L, L M, M G, be drawn touching † the Circle in the Points † 17. 3.
A, B, C, D, E: Let F be the Centre of the Circle A B C D E, and join F B, F K, F C, F L, F D.

Then, becauſe the Right Line K L touches the Circle A B C D E in the Point C, and the Right Line F C is drawn from the Centre F to C, the Point of Contact; F C will be ‡ perpendicular to K L: And ſo ‡ 18. 3.
both the Angles at C are Right Angles. For the ſame Reaſon, the Angles at the Points B, D, are Right Angles. And becauſe F C K is a Right Angle, the Square of F K will be * equal to the Squares of F C, * 47. 1.
C K: And for the ſame Reaſon, the Square of F K is equal to the Squares of F B, B K. Therefore the Squares of F C, C K, are equal to the Squares of F B, B K. But the Square of F C is equal to the Square of F B. Wherefore the Square of C K ſhall be equal to the Square B K; and ſo B K is equal to C K. And becauſe F B is equal to F C, and F K is common; the two Sides B F, F K, are equal to the two C F, F K, and the Baſe B K is equal to the Baſe K C; and ſo the Angle B F K ſhall be † equal to the Angle † 8. 1.
K F C, and the Angle B K F to the Angle F K C. Therefore the Angle B F C is double to the Angle K F C, and the Angle B K C double to the Angle F K C: For the ſame Reaſon, the Angle C F D is double to the Angle C F L, and the Angle C L D double to the Angle C L F. And becauſe the Circumference B C is equal to the Circumference C D; the Angle B F C ſhall be ‡ equal to the Angle C F D. ‡ 27. 3.
But the Angle B F C is double to the Angle K F C, and the Angle D F C double to L F C. Therefore the Angle K F C is equal to the Angle C F L. And ſo F K C, F L C, are two Triangles, having two Angles of the one equal to two Angles of the other, each to each, and one Side of the one equal to one Side of the other, *viz.* the common Side F C; wherefore they ſhall have † the other Sides of the one equal † 26. 1.
to the other Sides of the other; and the other Angle of the one equal to the other Angle of the other. Therefore the Right Line K C is equal to the Right Line C L, and the Angle F K C to the Angle F L C. And ſince K C is equal to C L, K L ſhall be double to K C. And by the ſame Reaſon, we prove that

HK is double to BK. Again, becauſe BK has been proved equal to KC, and KL the double to KC, as alſo HK the double of BK, HK ſhall be equal to KL. So likewiſe we prove that GH, GM, and ML, are each equal to HK, or KL. Therefore the Pentagon GHKLM is equilateral. I ſay alſo, it is equiangular; for becauſe the Angle FKC is equal to the Angle FLC; and the Angle HKL has been proved to be double to the Angle FKC; and alſo KLM double to FLC: Therefore the Angle HKL ſhall be equal to the Angle KLM. By the ſame Reaſon we demonſtrate, that every one of the Angles KHG, HGM, GML, is equal to the Angle HKL, or KLM. Therefore the five Angles, GHK, HKL, KLM, LMG, MGH, are equal between themſelves. And ſo the Pentagon GHKLM is equiangular, and it has been proved likewiſe to be equilateral, and deſcribed about the Circle ABCDE; *which was to be done.*

PROPOSITION XIII.

PROBLEM.

To deſcribe a Circle in an equilateral and equiangular Pentagon.

LET ABCDE be an equilateral and equiangular Pentagon. It is required to inſcribe a Circle in the ſame.

*9. 1. Biſect * the Angles BCD, CDE, by the Right Lines CF, DF; and from the Point F, wherein CF, DF, meet each other, let the Right Lines FB, FA, FE, be drawn. Now, becauſe BC is equal to CD, and CF is common, the two Sides BC, CF, are equal to the two Sides DC, CF; and the Angle BCF is equal to the Angle DCF. Therefore the Baſe BF is † equal to the Baſe FD; and the Triangle BFC equal to the Triangle DCF, and the other Angles of the one equal to the other Angles of the other, which are ſubtended by the equal Sides: Therefore the Angle CBF ſhall be equal to the Angle CDF. And becauſe the Angle CDE is double to the Angle CDF, and the Angle CDE is equal to the Angle † 4. 1.

Angle ABC, as alſo CDF equal to CBF; the Angle CBA will be double to the Angle CBF; and ſo the Angle ABF equal to the Angle CBF. Wherefore the Angle ABC is biſected by the Right Line BF. After the ſame manner we prove, that either of the Angles BAE or AED is biſected by the Right Lines AF, FE. From the Point F draw * FG, FH, FK, FL, FM, perpendicular to the Right Lines AB, BC, CD, DE, EA. Then, ſince the Angle HCF is equal to the Angle KCF; and the Right Angle FHC equal to the Right Angle FKC; the two Triangles FHC, FKC, ſhall have two Angles of the one equal to two Angles of the other, and one Side of the one equal to one Side of the other, *viz.* the Side FC common to each of them. And ſo the other Sides of the one will be † equal to the other Sides of the other, and the Perpendicular FH equal to the Perpendicular FK. In the ſame manner we demonſtrate, that FL, FM, or FG, is equal to FH, or FK. Therefore the five Right Lines FG, FH, FK, FL, FM, are equal to each other. And ſo a Circle deſcribed on the Centre F, with either of the Diſtances FG, FH, FK, FL, FM, will paſs thro' the other Points, and ſhall touch the Right Lines AB, BC, CD, DE, EA; ſince the Angles at G, H, K, L, M, are Right Angles: For if it does not touch them, but cuts them, a Right Line drawn from the Extremity of the Diameter of a Circle at Right Angles to the Diameter, will fall within the Circle; which is ‡ abſurd. Therefore a Circle deſcribed on the Centre F, with the Diſtance of any one of the Points G, H, K, L, M, will not cut the Right Lines AB, BC, CD, DE, EA; and ſo will neceſſarily touch them; *which was to be done.*

* 12. 1.

† 26. 1.

‡ 16. 3.

Coroll. If two of the neareſt Angles of an equilateral and equiangular Figure be biſected, and from the Point in which the Lines biſecting the Angles meet, there be drawn Right Lines to the other Angles of the Figure, all the Angles of the Figure will be biſected.

PRO-

PROPOSITION XIV.

PROBLEM.

To deſcribe a Circle about a given equilateral and equiangular Pentagon.

LET ABCDE be an equilateral and equiangular Pentagon. It is required to deſcribe a Circle about the ſame.

Biſect both the Angles BCD, CDE, by the Right Lines CF, FD, and draw FB, FA, FE, from the Point F, in which they meet. Then each of the Angles CBA, BAE, AED, ſhall be biſected * by the Right Lines BF, FA, FE. And ſince the Angle BCD is equal to the Angle CDE; and the Angle FCD is half the Angle BCD, as likewiſe CDF, half CDE; the Angle FCD will be equal to the Angle FDC; and ſo the Side CF †, equal to the Side FD. We demonſtrate in like manner, that FB, FA, or FE, is equal to FC, or FD. Therefore the five Right Lines FA, FB, FC, FD, FE, are equal to each other. And ſo a Circle being deſcribed on the Centre F, with any of the Diſtances FA, FB, FC, FD, FE, will paſs through the other Points, and will be deſcribed about the equilateral and equiangular Pentagon ABCDE; *which was to be done.*

* *Cor. of Preced.*

† 6. 1.

PROPOSITION XV.

PROBLEM.

To inſcribe an equilateral and equiangular Hexagon in a given Circle.

LET ABCDEF be a Circle given. It is required to inſcribe an equilateral and equiangular Hexagon therein.

Draw AD a Diameter of the Circle ABCDEF, and let G be the Centre; and about the Point D, as a Centre, with the Diſtance DG, let a Circle EGCH, be deſcribed; join EG, GC, which produce to the Points B, F: Likewiſe join AB, BC, CD, DE, EF, FA. I ſay, ABCDEF is an equilateral and equiangular Hexagon.

For, ſince the Point G is the Centre of the Circle ABCDEF, GE will be equal to GD. Again, becauſe the Point D is the Centre of the Circle EGCH, DE ſhall be equal to DG: But GE has been proved equal to GD. Therefore GE is equal to ED. And ſo EGD is an equilateral Triangle; and conſequently the three Angles thereof, EGD, GDE, DEG, are * equal between themſelves: But the * *Cor.* 5. 1. three Angles of a Triangle are † equal to two Right † 32. 1. Angles. Therefore the Angle EGD is a third Part of two Right Angles. In the ſame manner we demonſtrate, that DGC is one third Part of two Right Angles: And ſince the Right Line CG, ſtanding upon the Right Line EB, makes ‡ the adjacent Angles ‡ 13. 1. EGC, CGB; the other Angle, CGB, is alſo one third Part of two Right Angles. Therefore the Angles EGD, DGC, CGB, are equal between themſelves: And the Angles that are vertical to them, *viz.* the Angles BGA, AGF, FGE, are * equal to the * 15. 1. Angles EGD, DGC, CGB. Wherefore the ſix Angles EGD, DGC, CGB, BGA, AGF, FGE, are equal to one another. But equal Angles ſtand † on † 26. 3. equal Circumferences. Therefore the ſix Circumferences, AB, BC, CD, DE, EF, FA, are equal to each other. But equal Right Lines ſubtend ‡ equal ‡ 29. 3. Circumferences. Therefore the ſix Right Lines are equal between themſelves; and accordingly the Hexagon ABCDEF is equilateral: I ſay, it is alſo equiangular. For, becauſe the Circumference AF is equal to the Circumference ED, add the common Circumference ABCD, and the whole Circumference FABCD, is equal to the whole Circumference EDCBA. But the Angle FED ſtands on the Circumference FABCD; and the Angle AFE, on the Circumference EDCBA. Therefore the Angle AFE is * equal to the Angle DEF. In the ſame * 27. 3. manner we prove, that the other Angles of the Hexagon ABCDEF, are ſeverally equal to AFE, or FED. Therefore the Hexagon ABCDEF is equiangular. But it has been proved to be alſo equilateral, and is inſcribed in the Circle ABCDEF; *which was to be done.*

Coroll. From hence it is manifeſt, that the Side of the Hexagon is equal to the Semidiameter of the Circle. And if we draw thro' the Points A, B, C, D, E, F, Tangents to the Circle, an equilateral and equiangular Hexagon will be deſcribed about the Circle, as is manifeſt from what has been ſaid concerning the Pentagon. And ſo likewiſe may a Circle be inſcribed and circumſcribed about a given Hexagon; *which was to be done.*

PROPOSITION XVI.

PROBLEM.

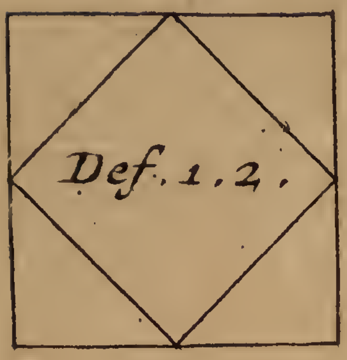
To deſcribe an equilateral and equiangular Quindecagon in a given Circle.

LET ABCD be a Circle given. It is required to deſcribe an equilateral and equiangular Quindecagon in the ſame.

Let AC be the Side of an equilateral Triangle inſcribed in the Circle ABCD, and AB the Side of a Pentagon. Now, if the whole Circumference of the Circle ABCD be divided into fifteen equal Parts, the Circumference ABC, one Third of the whole, ſhall be five of the ſaid fifteen equal Parts; and the Circumference AB, one Fifth of the Whole, will be three of the ſaid Parts. Wherefore the remaining Circumference BC will be two of the ſaid Parts. And if BC be biſected in the Point E, BE, or EC, will be one fifteenth Part of the whole Circumference ABCD. And ſo, if BE, EC, be joined, and either EC, or EB, be continually applied in the Circle, there ſhall be an equilateral and equiangular Quindecagon deſcribed in the Circle ABCD; *which was to be done.*

If, according to what has been ſaid of the Pentagon, Right Lines are drawn thro' the Diviſions of the Circle touching the ſame, there will be deſcribed about the Circle an equilateral and equiangular Quindecagon. And, moreover, a Circle may be inſcribed, or circumſcribed, about a given equilateral and equiangular Quindecagon.

Book 4.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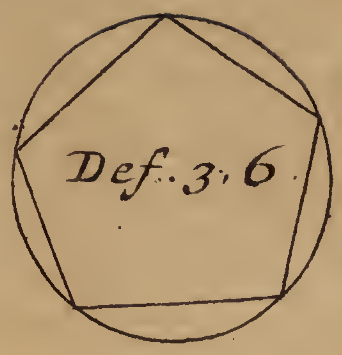
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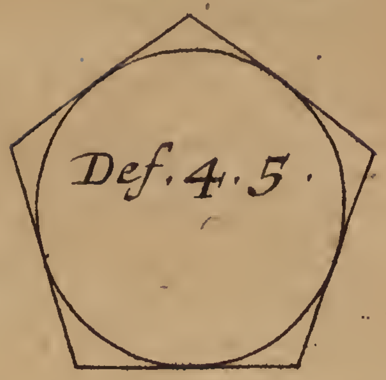
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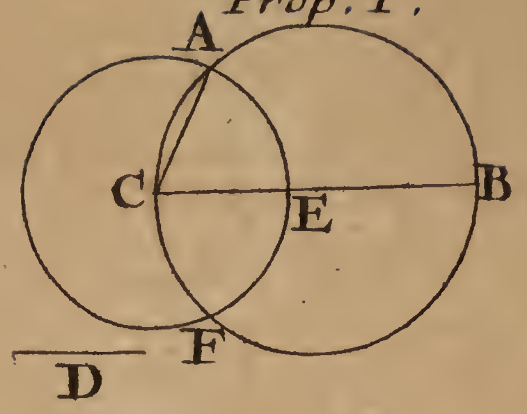

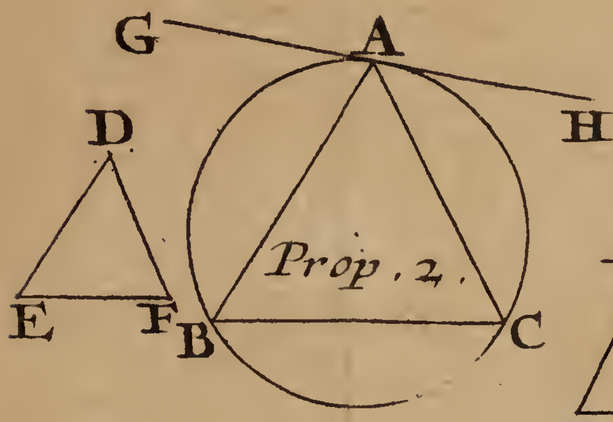

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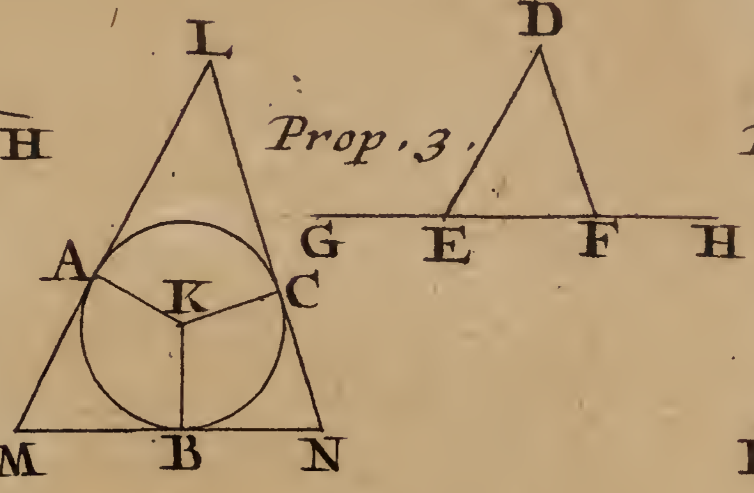

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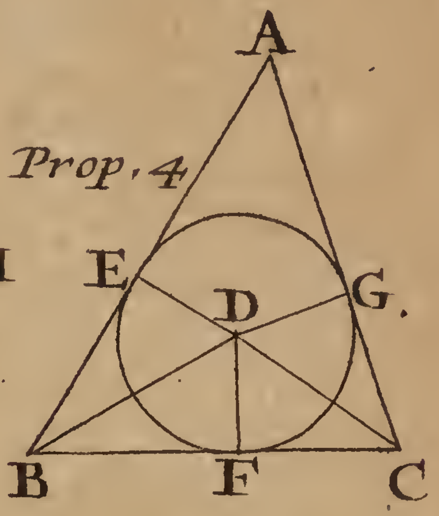

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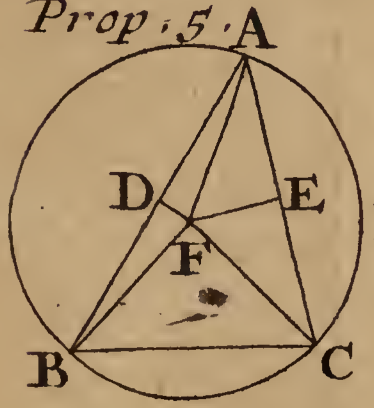

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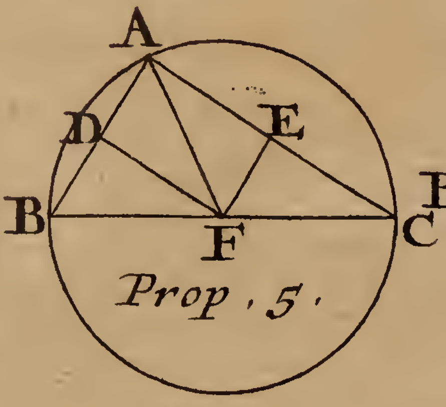

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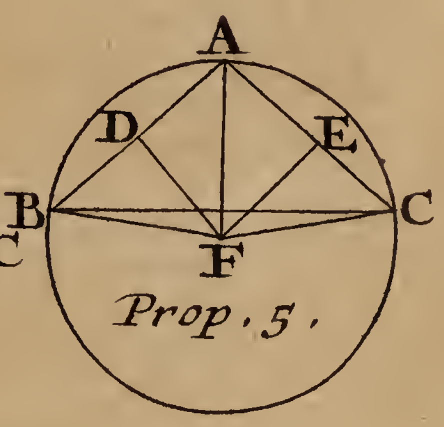

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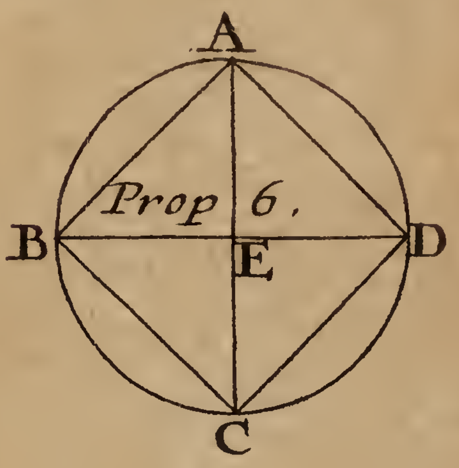

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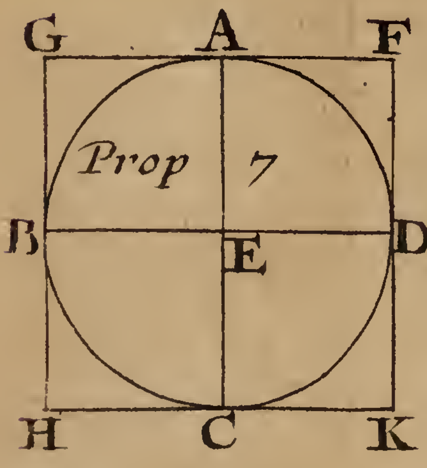

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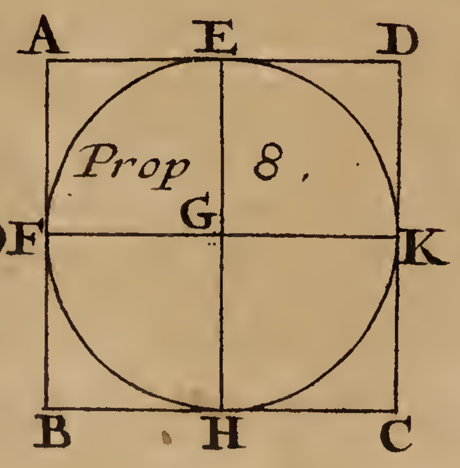

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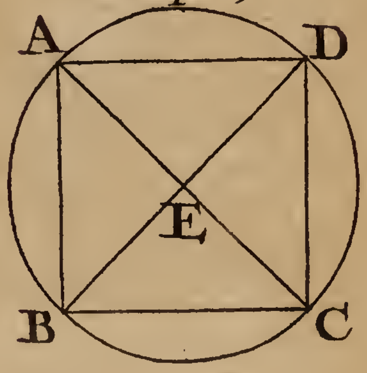

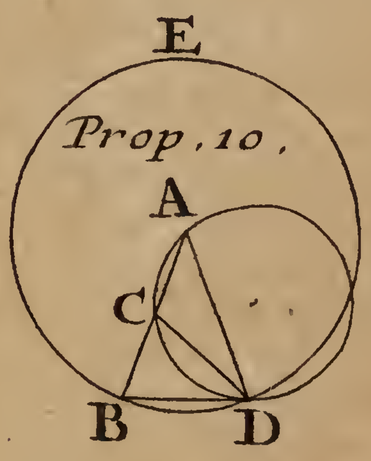

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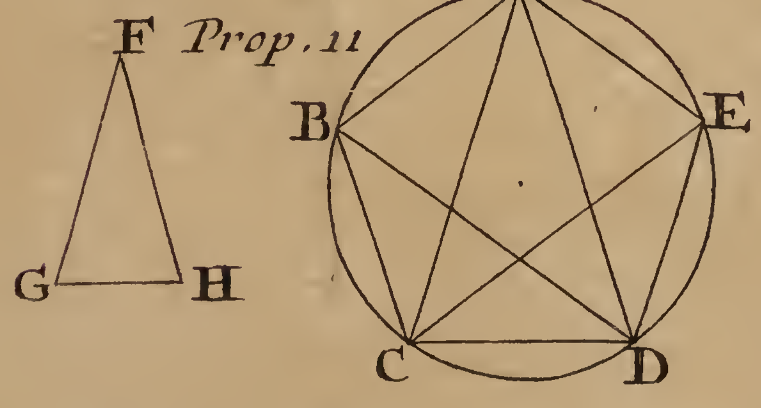

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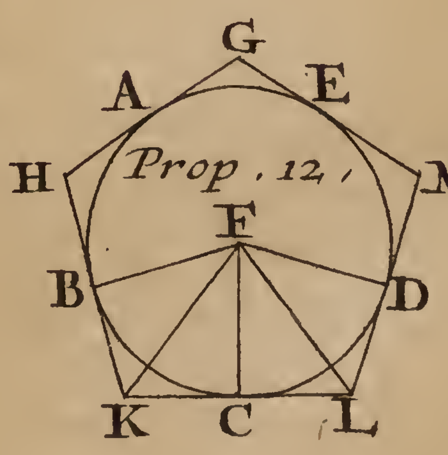

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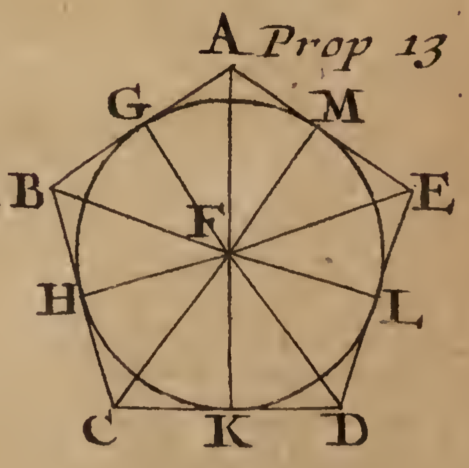

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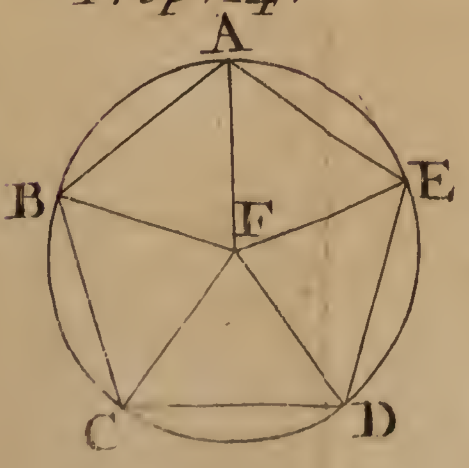

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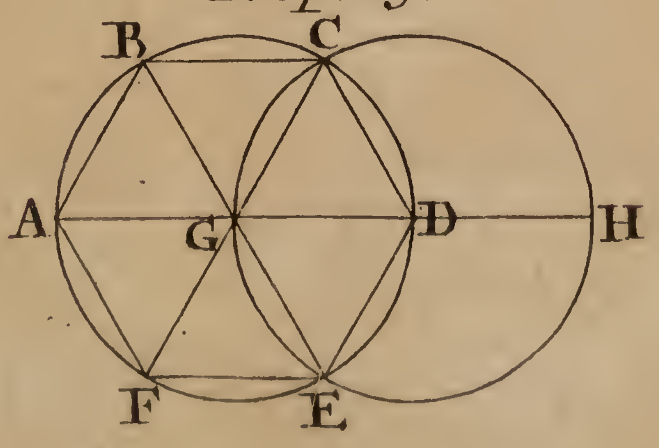
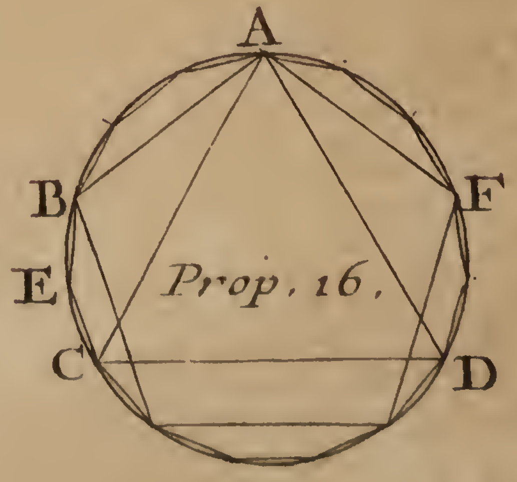




Prop. 16.

EUCLID's ELEMENTS.

BOOK V.

DEFINITIONS.

I. *A PART is a Magnitude of a Magnitude, the Leſs of the Greater, when the Leſſer meaſures the Greater.* * Aliquot

II. *But a Multiple is a Magnitude of a Magnitude, the Greater of the Leſſer, when the Leſſer meaſures the Greater.*

III. *Ratio is a certain mutual Habitude of Magnitudes of the ſame Kind, according to Quantity.*

IV. *Magnitudes are ſaid to have Proportion to each other, which being multiplied can exceed one another.*

V. *Magnitudes are ſaid to be in the ſame Ratio, the firſt to the ſecond, and the third to the fourth, when the Equimultiples of the firſt and third, compared with the Equimultiples of the ſecond and fourth, according to any Multiplication whatſoever, are either both together greater, equal, or leſs than the Equimultiples of the ſecond and fourth, if thoſe be taken that anſwer each other.*

That is, if there be four Magnitudes, and you take any Equimultiples of the firſt and third, and alſo any Equimultiples of the ſecond and fourth; and if the Multiple of the firſt be greater than the Multiple of the ſecond, and alſo the Multiple of the third greater than the Multiple of the fourth: Or, if the Multiple of the firſt be equal to the Multiple of the ſecond; and alſo the Multiple of the third equal to the Multiple of the fourth: Or, laſtly, if the Multiple of the firſt be leſs than the Multiple of the ſecond; and alſo that of the third leſs than that of the fourth, and theſe Things happen according to every Multiplication whatſoever; then the four Magnitudes are in the ſame Ratio, the firſt to the ſecond, as the third to the fourth.

VI. *Magnitudes that have the ſame Proportion are called Proportionals.*

Expounders uſually lay down here that Definition which *Euclid* has given for Numbers only, in his ſeventh Book; *viz.* That

Magnitudes are ſaid to be Proportionals, when the firſt is the ſame Equimultiple of the ſecond, as the third is of the fourth, or the ſame Part or Parts.

But this Definition appertains only to Numbers, and commenſurable Quantities; and ſo, ſince it is not univerſal, *Euclid* did well to reject it in this Element, which treats of the Properties of all Proportionals; and to ſubſtitute another general one, agreeing to all Kinds of Magnitudes. In the mean time, Expounders very much endeavour to demonſtrate the Definition here laid down by *Euclid*, by the uſual received Definition of proportional Numbers; but this much eaſier flows from that, than that from this; which may be thus demonſtrated:

Firſt, Let A, B, C, D, be four Magnitudes, which are in the ſame Ratio, according to the Conditions that Magnitudes in the ſame Ratio muſt have laid down in the fifth Definition. And let the firſt be a Multiple of the ſecond. I ſay, the third is alſo the ſame Multiple of the fourth. For Example: Let A be equal to 5B. Then C ſhall be equal to 5D. Take any Number; for Example, 2, by which let 5 be multiplied, and the Product will be 10: And let 2A, 2C, be Equimultiples of the firſt and third Magnitudes A and C: Alſo, let 10B and 10D be Equimultiples of the ſecond and fourth Magnitudes B and D. Then (by *Def.* 5.) if 2A be equal to 10B, 2C ſhall be equal to 10D. But ſince A (from the *Hypotheſis*) is five times B, 2A ſhall be equal to 10B; and ſo 2C equal to 10D, and C equal to 5D; that is, C will be five times D. W. W. D.

A: B:: C: D

2A, 10B, 2C, 10D

Secondly, Let A be any Part of B; then C will be the ſame Part of D. For, becauſe A is to B, as C is to D; and ſince A is ſome Part of B; then B will be a Multiple of A: And ſo (by *Caſe* 1.) D will be the ſame Multiple of C, and accordingly C ſhall be the ſame Part of the Magnitude D, as A is of B. W. W. D.

Thirdly, Let A be equal to any Number of whatſoever Parts of B. I ſay, C is equal to the ſame Number of the like Parts of D. For Example: Let A be a fourth Part of five times B; that is, let A be equal to $\frac{5}{4}$B. I ſay, C is alſo equal to $\frac{5}{4}$D. For, becauſe A is equal to $\frac{5}{4}$B, each of them being multiplied by 4, then 4A will be equal to 5B. And ſo, if the Equimultiples of the firſt and third, *viz.* 4A, 4C, be aſſumed; as alſo the Equimultiples of the ſecond and fourth, *viz.* 5B, 5D, and (by the *Definition*) if 4A is equal to 5B; then 4C is equal to 5D. But 4A has been proved equal to 5B, and ſo 4C ſhall be equal to 5D, and C equal to $\frac{5}{4}$D. W. W. D.

A: B:: C: D

4A, 5B, 4C, 5D

And univerſally, if A be equal to $\frac{n}{m}$B, C will be equal to $\frac{n}{m}$D. For let A and C be multiplied by *m*, and B and D by *n*. And becauſe A is equal to $\frac{n}{m}$B; *m*A ſhall be equal to *n*B; wherefore (by *Def.* 5.) *m*C will be equal to *n*D, and C equal to $\frac{n}{m}$D. W. W. D.

A: B:: C: D

*m*A, *n*B, *m*C, *n*D

VII. *When of Equimultiples, the Multiple of the firſt exceeds the Multiple of the ſecond; but the Multiple of the third does not exceed the Multiple of the fourth; then the firſt to the ſecond is ſaid to have a greater Proportion, than the third to the fourth.*

VIII. *Analogy is a Similitude of Proportions.*

IX. *Analogy at leaſt conſiſts of three Terms.*

* X. *When three Magnitudes are Proportionals, the firſt is ſaid to have, to the third, a Duplicate Ratio to what it has to the ſecond.*

* XI. *But when four Magnitudes are Proportionals, the firſt ſhall have a triplicate Ratio to the fourth of what it has to the ſecond; and ſo always one more in Order, as the Proportionals ſhall be extended.*

* XII. *Homologous Magnitudes, or Magnitudes of a like Ratio, are ſaid to be ſuch whoſe Antecedents are to the Antecedents, and Conſequents to the Conſequents.*

XIII. *Alternate Ratio is the comparing of the Antecedent with the Antecedent, and the Conſequent with the Conſequent.*

XIV. *Inverſe Ratio is when the Conſequent is taken as the Antecedent, and ſo compared with the Antecedent as a Conſequent.*

XV. *Compounded Ratio is when the Antecedent and Conſequent taken both as one, is compared to the Conſequent itſelf.*

XVI. *Divided Ratio is when the Exceſs, wherein the Antecedent exceeds the Conſequent, is compared with the Conſequent.*

XVII. *Converſe Ratio is when the Antecedent is compared with the Exceſs, by which the Antecedent exceeds the Conſequent.*

XVIII. *Ratio of Equality is where there are taken more than two Magnitudes in one Order, and a like*

like Number of Magnitudes in another Order, comparing two to two being in the ſame Proportion; and it ſhall be in the firſt Order of Magnitude, as the firſt is to the laſt, ſo in the ſecond Order of Magnitudes is the firſt to the laſt: Or otherwiſe, it is the Compariſon of the Extremes together, the Means being omitted.

XIX. *Ordinate Proportion is when, as the Antecent is to the Conſequent, ſo is the Antecedent to the Conſequent; and as the Conſequent is to any other, ſo is the Conſequent to any other.*

XX. *Perturbate Proportion is when there are three Magnitudes, and others alſo, that are equal to theſe in Multitude, as in the firſt Magnitudes the Antecedent is to the Conſequent; ſo in the ſecond Magnitude is the Antecedent to the Conſequent: And as in the firſt Magnitudes the Conſequent is to ſome other, ſo in the ſecond Magnitudes, is ſome other to the Antecedent.*

AXIOMS.

I. *EQUIMULTIPLES of the ſame, or of equal Magnitudes, are equal to each other.*

II. *Thoſe Magnitudes that have the ſame Equimultiple, or whoſe Equimultiples are equal, are equal to each other.*

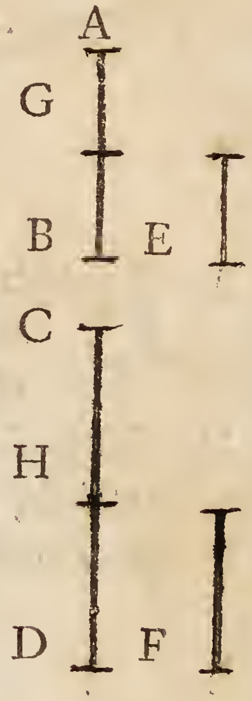
PROPOSITION I.

THEOREM.

If there be any Number of Magnitudes Equimultiples of a like Number of Magnitudes, each to each; whatſoever Multiple any one of the former Magnitudes is of its correſpondent one, the ſame Multiple is all the former Magnitudes of, all the latter.

LET there be any Number of Magnitudes AB, CD, Equimultiples of a like Number of Magnitudes E, F, each of each. I ſay, what Multiple the Magnitude AB is of E, the ſame Multiple AB, and CD, together, is of E and F together.

For, becauſe AB and CD are Equimultiples of E and F, as many Magnitudes equal to E, that are in AB, ſo many ſhall be equal to F in CD. Now, divide AB into Parts equal to E, which let be AG, GB; and CD into Parts equal to F, *viz.* CH, HD. Then the Multitude of Parts, CH, HD, ſhall be equal to the Multitude of Parts AG, GB. And ſince AG is equal to E, and CH to F; AG and CH, together, ſhall be equal to E and F together. By the ſame Reaſon, becauſe GB is equal to E, and HD to F, GB and HD will be equal to E and F together. Therefore, as often as E is contained in AB, ſo often is E and F contained in AB and CD. And ſo, as often as E is contained in AB, ſo often are E and F, together, contained in AB and CD together. Therefore, *if there are any Number of Magnitudes Equimultiples of a like Number of Magnitudes, each to each; whatſoever Multiple any one of the former Magnitudes is of its Correſpondent one, the ſame Multiple is all the former Magnitudes of, all the latter;* which was to be demonſtrated.



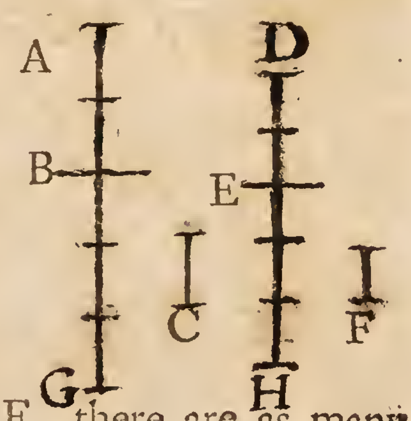
PRO-

PROPOSITION II.

THEOREM.

If the firſt be the ſame Multiple of the ſecond, as the third is of the fourth; and if the fifth be the ſame Multiple of the ſecond, as the ſixth is of the fourth; then ſhall the firſt, added to the fifth, be the ſame Multiple of the ſecond, as the third, added to the ſixth, is of the fourth.

LET the firſt AB be the ſame Multiple of the ſecond C, as the third DE is of the fourth F; and let the fifth BG be the ſame Multiple of the ſecond C, as the ſixth EH is of the fourth F. I ſay, the firſt added to the fifth, *viz.* AG, is the ſame Multiple of the ſecond C, as the third added to the ſixth, *viz.* DH, is of the fourth F.



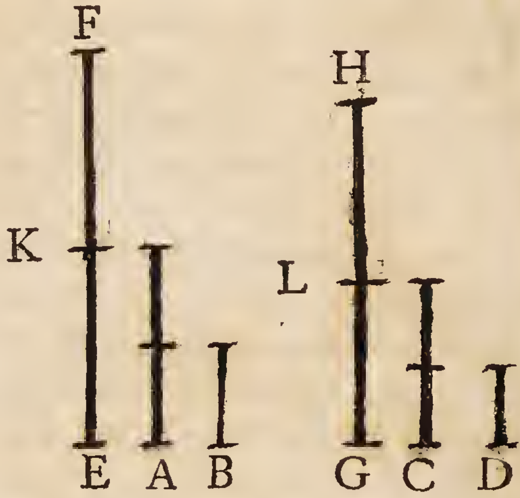
For, becauſe AB is the ſame Multiple of C, as DE is of F, there are as many Magnitudes equal to C in AB, as there are Magnitudes equal to F in DE. And for the ſame Reaſon, there are as many Magnitudes equal to C in BG, as there are Magnitudes equal to F in EH. Therefore there are as many Magnitudes equal to C, in the whole AG, as there are Magnitudes equal to F in DH. Wherefore AG is the ſame Multiple of C, as DH is of F. And ſo the firſt added to the fifth AG, is the ſame Multiple of the ſecond C, as the third, added to the ſixth DH, is of the fourth F. Therefore, *if the firſt be the ſame Multiple of the ſecond, as the third is of the fourth; and if the fifth be the ſame Multiple of the ſecond, as the ſixth is of the fourth; then ſhall the firſt, added to the fifth, be the ſame Multiple of the ſecond, as the third, added to the ſixth, is of the fourth;* which was to be demonſtrated.

PROPOSITION III.

THEOREM.

If the firſt be the ſame Multiple of the ſecond, as the third is of the fourth, and there be taken Equimultiples of the firſt and third; then will each of the Magnitudes taken be Equimultiples of the ſecond and fourth.

LET the firſt A be the ſame Multiple of the ſecond B, as the third C is of the fourth D; and let EF, GH, be Equimultiples of A and C. I ſay, EF is the ſame Multiple of B, as GH is of D.



For, becauſe EF is the ſame Multiple of A, as GH is of C, there are as many Magnitudes equal to A in EF, as there are Magnitudes equal to C in GH. Now, divide EF into the Magnitudes EK, KF, equal to A, and GH into the Magnitudes GL, LH, equal to C. Then the Number of the Magnitudes EK, KF, will be equal to the Number of the Magnitudes GL, LH. And becauſe A is the ſame Multiple of B, as C is of D, and EK is equal to A, and GL to C; EK will be the ſame Multiple of B, as GL is of D. For the ſame Reaſon, KF ſhall be the ſame Multiple of B, as LH is of D. Therefore, becauſe the firſt EK is the ſame Multiple of the ſecond B, as the third GL is of the fourth D, and KF, LH, are Equimultiples of the ſecond B and fourth D; the firſt added to the fifth EF, ſhall be * the ſame Multiple of the ſecond B, as the third added to the ſixth GH is of the fourth D. *If*, therefore, *the firſt be the ſame Multiple of the ſecond, as the third is of the fourth, and there be taken Equimultiples of the firſt and third; then will each of the Magnitudes taken be Equimultiples of the ſecond and fourth*; which was to be demonſtrated.

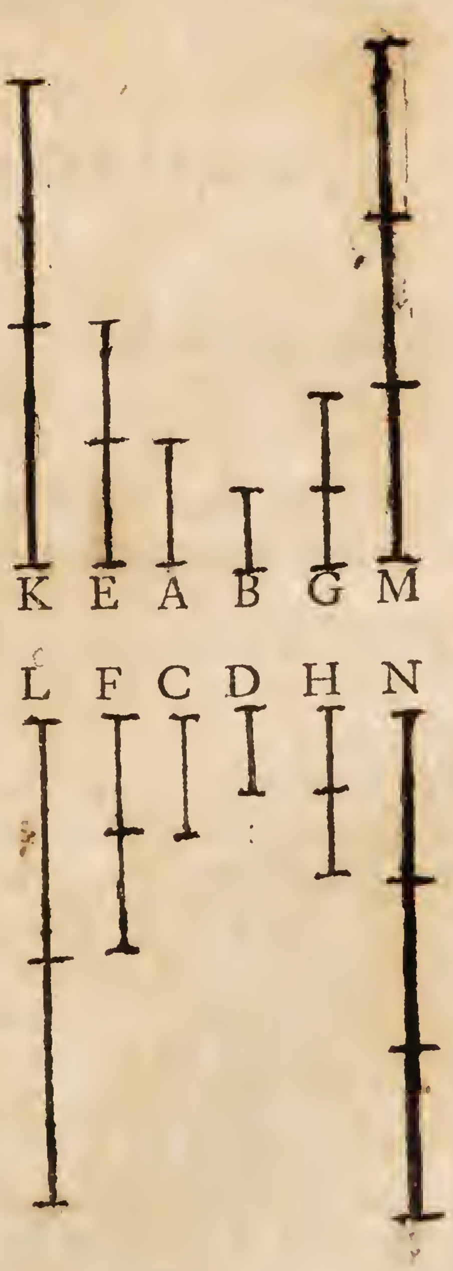
* *2 of this.*

PROPOSITION IV.

THEOREM.

If the firſt have the ſame Proportion to the ſecond, as the third to the fourth; then alſo ſhall the Equimultiples of the firſt and third have the ſame Proportion to the Equimultiples of the ſecond and fourth, according to any Multiplication whatſoever, if they be ſo taken as to anſwer each other.

LET the firſt A have the ſame Proportion to the ſecond B, as the third C hath to the fourth D; and let E and F, the Equimultiples of A and C, be any-how taken; as alſo G, H, the Equimultiples of B and D. I ſay, E is to G as F is to H.



For take K and L, any Equimultiples of E and F; and alſo M and N of G and H.

Then, becauſe E is the ſame Multiple of A, as F is of C, and K, L, are taken Equimultiples of E, F, K will be * the ſame Multiple of A, as L is of C. For the ſame Reaſon, M is the ſame Multiple of B, as N is of D. And ſince A is to B, as C is to D, and K and L are Equimultiples of A and C; and alſo M and N Equimultiples of B and D; if K exceeds M, then † L will exceed N; if equal, equal; or leſs, leſs. And K, L, are Equimultiples of E, F, and M, N, any other Equimultiples of G H. Therefore, as E is to G, ſo

* *3 of this.*

† *Def. 5. of this.*

ſhall

‡ *Def.* 5. ſhall ‡ F be to H. Wherefore, *if the firſt have the ſame Proportion to the ſecond, as the third to the fourth; then alſo ſhall the Equimultiples of the firſt and third have the ſame Proportion to the Equimultiples of the ſecond and fourth, according to any Multiplication whatſoever, if they be ſo taken as to anſwer each*; which was to be demonſtrated.

Becauſe it is demonſtrated, if K exceeds M, then L will exceed N; and if it be equal to it, it will be equal; and if leſs, leſſer. It is manifeſt likewiſe, if M exceeds K, that N ſhall exceed L; if equal, equal; but if leſs, leſs. And therefore as G is to E, ſo is * H to F. * *Def.* 5.

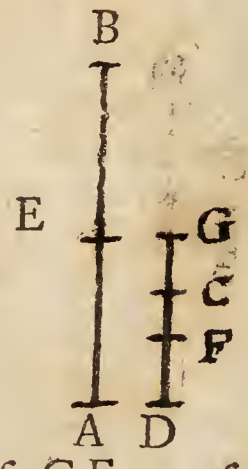
Coroll. From hence it is manifeſt, if four Magnitudes be proportional, that they will be alſo inverſely proportional.

PROPOSITION V.

THEOREM.

If one Magnitude be the ſame Multiple of another Magnitude, as a Part taken from the one is of a Part taken from the other; then the Reſidue of the one ſhall be the ſame Multiple of the Reſidue of the other, as the Whole is of the Whole.

LET the Magnitude AB be the ſame Multiple of the Magnitude CD, as the Part taken away AE is of the Part taken away CF. I ſay, that the Reſidue EB is the ſame Multiple of the Reſidue FD, as the whole AB is of the whole CD.



For, let EB be ſuch a Multiple of CG, as AE is of CF.

Then, becauſe AE is the ſame Multiple of CF, as EB is of CG, AE will be * the ſame Multiple of CF, as AB is of GF. But AE and AB are put Equimultiples of CF and CD. Therefore AB is the ſame Multiple of GF as of CD; and ſo GF is † equal to CD. Now, let CF, which is common, be taken away; and the Reſidue

* 1 *of this.*

† 2 *Axiom of this.*

GC

G C is equal to the Reſidue D F. And then, becauſe A E is the ſame Multiple of C F, as E B is of C G, and C G is equal to D F; A E ſhall be the ſame Multiple of C F, as E B is of F D. But A E is put the ſame Multiple of C F, as A B is of C D. Therefore E B is the ſame Multiple of F D, as A B is of C D: and ſo the Reſidue E B is the ſame Multiple of the Reſidue F D, as the whole A B is of the whole C D. Wherefore, *if one Magnitude be the ſame Multiple of another Magnitude, as a Part taken from the one is of a Part taken from the other; then the Reſidue of the one ſhall be the ſame Multiple of the Reſidue of the other, as the whole is of the whole*; which was to be demonſtrated.

PROPOSITION VI.

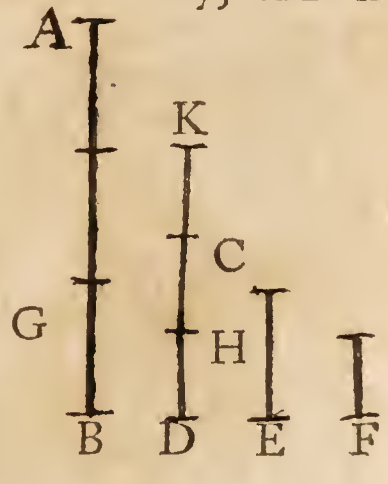
THEOREM.

If two Magnitudes be Equimultiples of two Magnitudes, and ſome Magnitudes, Equimultiples of the ſame, be taken away; then the Reſidues are either equal to thoſe Magnitudes, or elſe Equimultiples of them.

LET two Magnitudes A B, C D, be Equimultiples of two Magnitudes E, F, and let the Magnitudes A G, C H, Equimultiples of the ſame E, F, be taken from A B, C D. I ſay, the Reſidues G B, H D, are either equal to E, F, or are Equimultiples of them.

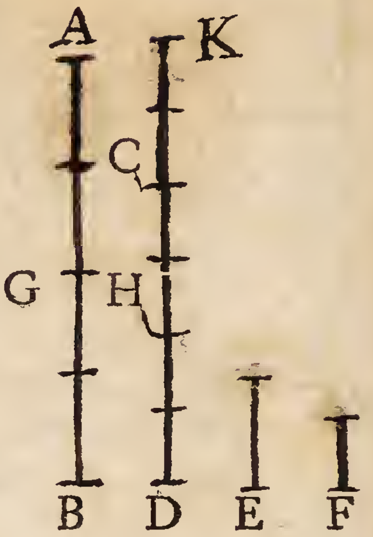
For firſt, let G B be equal to E. I ſay, H D is alſo equal to F. For let C K be equal to F. Then, becauſe A G is the ſame Multiple of E, as C H is of F; and G B is equal to E; and C K to F; AB will be * the ſame Multiple of E, as K H is of F. But A B and C D are put Equimultiples of E and F. Therefore K H is the ſame Multiple of F, as C D is of F.

* 1 of this.



And becauſe K H and C D are Equimultiples of F; K H will be equal to C D.

Take



Take away CH, which is common; then the Reſidue KC is equal to the Reſidue HD. But KC is equal to F. Therefore HD is equal to F; and ſo GB ſhall be equal to E, and HD to F.

In like manner we demonſtrate, if GB was a Multiple of E, that HD is the like Multiple of F. Therefore, *if two Magnitudes be Equimultiples of two Magnitudes, and ſome Magnitudes, Equimultiples of the ſame, be taken away; then the Reſidues are either equal to thoſe Magnitudes, or elſe Equimultiples of them*; which was to be demonſtrated.

PROPOSITION VII.

PROBLEM.

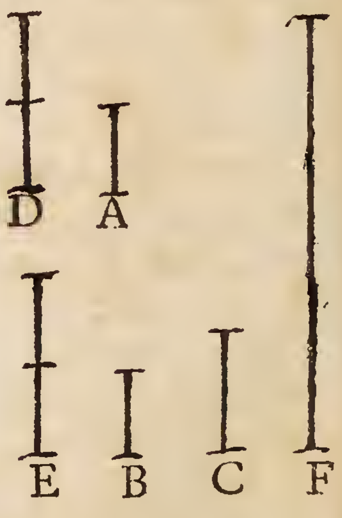
Equal Magnitudes have the ſame Proportion to the ſame Magnitude; and one and the ſame Magnitude has the ſame Proportion to equal Magnitudes.

LET A, B, be equal Magnitudes, and let C be any other Magnitude. I ſay, A and B have the ſame Proportion to C; and likewiſe C has the ſame Proportion to A as to B.

For take D, E, Equimultiples of A and B; and let F be any other Multiple of C.

Now, becauſe D is the ſame Multiple of A, as E is of B, and A is equal to B, D ſhall be alſo equal to E; but F is a Magnitude taken at Pleaſure. Therefore, if D exceeds F, then E will exceed F; if D be equal to F, E will be equal to F; and if leſs, leſs. But D, E, are Equimultiples of A, B; and F is any Multiple of C. Therefore

* Def. 5. it will be *as A is to C, ſo is B to C.



I ſay, moreover, that C has the ſame Proportion to A as to B. For the ſame Conſtruction remaining, we prove, in like manner, that D is equal to E. Therefore, if F exceeds D, it will alſo exceed E; if it be equal to D, it will be equal to E; and if it be leſs than D, it will be leſs than E. But if F is Multiple of C; and D, E, any other Equimultiples of A, B. Therefore as C is to A, ſo ſhall *C be to B. Wherefore equal Magnitudes have the ſame Proportion to the ſame Magnitude, and the ſame Magnitude to equal ones; *which was to be demonſtrated.*

** Def. 5.*

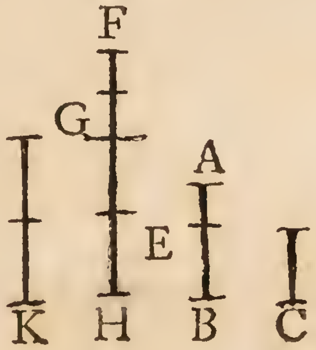
PROPOSITION VIII.

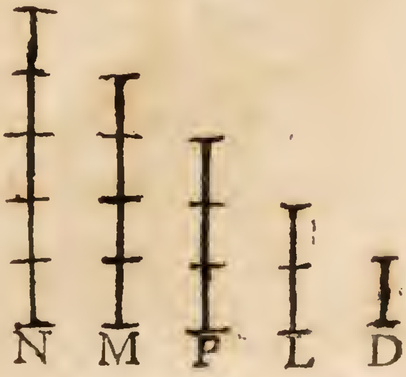
THEOREM.

The greater of any two unequal Magnitudes has a greater Proportion to ſome third Magnitude, than the leſs has; and that third Magnitude hath a greater Proportion to the leſſer of the two Magnitudes, than it has to the greater.

LET AB and C be two unequal Magnitudes, whereof AB is the greater: and let D be any third Magnitude. I ſay, AB has a greater Proportion to D, than C has to D; and D has a greater Proportion to C, than it has to AB.

Becauſe AB is greater than C, make BE equal to C, that is, let AB exceed C by AE; then AE multiplied ſome Number of Times, will be greater than D. Now let AE be multiplied until it exceeds D, and let that Multiple of AE, greater than D, be FG. Make GH the ſame Multiple of EB, and K of C, as FG is of AE. Alſo, aſſume L double to D, P triple, and ſo on, until ſuch a Multiple of D is had, as is the neareſt greater than K; let this be N, and let M be a Multiple of D the neareſt leſs than N.





Now, becauſe N is the neareſt Multiple of D greater than K, M will not be greater than K; that is, K will not be leſs then M. And ſince FG is the ſame Multiple of AE, as GH is of EB; FG ſhall be * the ſame Multiple of AE, as FH is of AB; but FG is the ſame Multiple of AE, as K is of C; wherefore FH is the ſame Multiple of AB, as K is of C; that is, FH, K, are Equimultiples of AB and C. Again, becauſe GH is the ſame Multiple of EB, as K is of C, and EB is equal to C; GH ſhall be † equal to K. But K is not leſs than M. Therefore GH ſhall not be leſs than M; but FG is greater than D. Therefore the whole FH will be greater than M and D; but M and D together, are equal to N; becauſe M is a Multiple of D, the neareſt leſſer than N: Wherefore FH is greater than N. And ſo, ſince FH exceeds N, and K does not, and FH and K are Equimultiples of AB and C, and N is another Multiple of D; therefore AB will have ‡ a greater Ratio to D, than C has to D. I ſay, moreover, that D has a greater Ratio to C, than it has to AB; for the ſame Conſtruction remaining, we demonſtrate, as before, that N exceeds K, but not FH. And N is a Multiple of D, and FH, K, are Equimultiples of AB and C. Therefore D has ‡ a greater Proportion to C, than D hath to B. Wherefore *the greater of any two unequal Magnitudes has a greater Proportion to ſome third Magnitude, than the leſs has; and that third Magnitude hath a greater Proportion to the leſſer of the two Magnitudes, than it has to the greater.*

* 1 *of this.*

† *Ax.* 1.

‡ *Def.* 7.

PROPOSITION IX.

THEOREM.

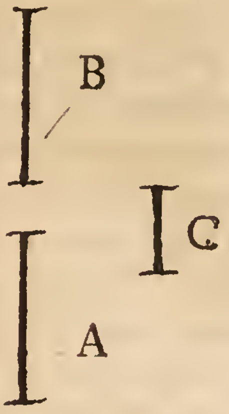
Magnitudes which have the ſame Proportion to one and the ſame Magnitude, are equal to one another; and if a Magnitude has the ſame Proportion to other Magnitudes, theſe Magnitudes are equal to one another.

LET the Magnitudes A and B have the ſame Proportion to C. I ſay, A is equal to B.

For if it was not, A and B would not * have the ſame Proportion to the ſame Magnitude C; but they have. Therefore A is equal to B. * 8 *of this.*

Again, let C have the ſame Proportion to A as to B. I ſay, A is equal to B.

For if it be not, C will not have the ſame Proportion to A as to B; but it hath: Therefore A is neceſſarily equal to B. Therefore, *Magnitudes that have the ſame Proportion to one and the ſame Magnitude, are equal to one another; and if a Magnitude has the ſame Proportion to other Magnitudes, theſe Magnitudes are equal to one another;* which was to be demonſtrated.



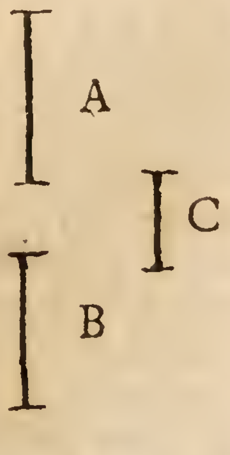
PROPOSITION X.

THEOREM.

Of Magnitudes having Proportion to the ſame Magnitude, that which has the greater Proportion, is the greater Magnitude: And the Magnitude to which the ſame bears a greater Proportion, is the leſſer Magnitude.

LET A have a greater Proportion to C, than B has to C. I ſay, A is greater than B.

For if it be not greater, it will either be equal or leſs. But A is not equal to B, becauſe then both A and B would have * the ſame Proportion to the ſame Magnitude C; but they have not. Therefore A is not equal to B: Neither is it leſs than B; for then A would have † a leſs Proportion to C, than B would have; but it hath not a leſs Proportion: Therefore A is not leſs than B. But it has been proved likewiſe not to be equal to it: Therefore A ſhall be greater than B. * 7 *of this.* † 8 *of this.*



Again, let C have a greater Proportion to B than to A. I ſay, B is leſs than A.

For if it be not leſs, it is greater or equal. Now
* 7 *of this.* B is not equal to A; for then C would have * the ſame Proportion to A as to B; but this it has not. Therefore A is not equal to B; neither is B greater than A; for if it was, C would have a leſs Proportion to B than to A; but it has not: Therefore B is not greater than A. But it has alſo been proved not to be equal to it. Wherefore B ſhall be leſs than A. Therefore *of Magnitudes having Proportion to the ſame Magnitude, that which has the greater Proportion, is the greater Magnitude: And that Magnitude to which the ſame bears a greater Proportion, is the leſſer Magnitude*; which was to be demonſtrated.

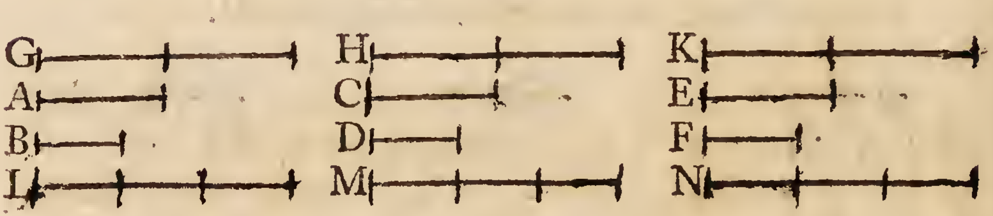
PROPOSITION XI.

THEOREM.

Proportions that are one and the ſame to any third, are alſo the ſame to one another.

LET A be to B, as C is to D; and C to D, as E to F. I ſay, A is to B, as E is to F.

For take G, H, K, Equimultiples of A, C, E; and L, M, N, other Equimultiples of B, D, F. Then, becauſe A is to B, as C is to D, and there are taken G, H, the Equimultiples of A and C, and L, M, any
* 5 *Def. of this.* Equimultiples of B, D; if G exceeds L, * then H will exceed M; and if G be equal to L, H will be equal to M; and if leſs, leſſer. Again, becauſe as C is to D, ſo is E to F; and H and K are taken Equimultiple of C and E; as likewiſe M, N, any Equimultiples of D, F; if H exceeds M*, then K will exceed N; and if H be equal to M, K will be equal to N; and if leſs, leſſer. But if H exceeds M, G will alſo exceed L; if equal, equal; and if leſs, leſs. Wherefore if G exceeds L, K will alſo exceed N; and if G be



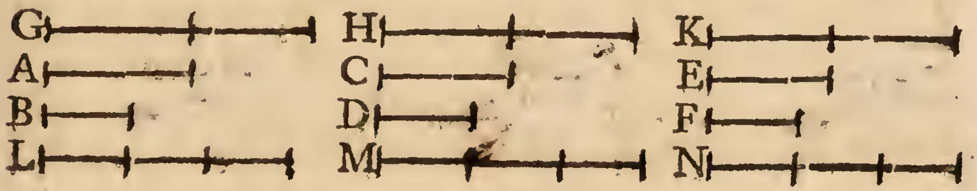
G be equal to L, K will be equal to N; and if leſs, leſs. But G, K, are Equimultiples of A, E; and L, N, are Equimultiples of B, F. Conſequently, as A is to B, ſo * is E to F. Therefore, *Proportions that are one and the ſame to any third, are alſo the ſame to one another*; which was to be demonſtrated. * 5 *Def. of this.*

PROPOSITION XII.

THEOREM.

If any Number of Magnitudes be proportional, as one of the Antecedents is to one of the Conſequents, ſo are all the Antecedents to all the Conſequents.

LET there be any Number of proportional Magnitudes, A, B, C, D, E, F; whereof as A is to B,



ſo C is to D, and ſo E to F. I ſay, as A is to B, ſo are all the Antecedents A, C, E, to all the Conſequents B, D, F.

For let G, H, K, be Equimultiples of A, C, E; and L, M, N, any Equimultiples of B, D, F.

Then, becauſe as A is to B, ſo is C to D, and ſo E to F; and G, H, K, are Equimultiples of A, C, E; and L, M, N, Equimultiples of B, D, F; if G exceeds L, H * will alſo exceed M, and K will exceed N; if G be equal to L, H will be equal to M, and K to N; and if leſs, leſs. Wherefore alſo, if G exceeds L, then G, H, K, together, will likewiſe exceed L, M, N, together; and if G be equal to L, then G, H, K, together, will be equal to L, M, N, together; and if leſs, leſs: But G, and G, H, K, are Equimultiples of A; and A, C, E; becauſe, if there are any Number of Magnitudes Equimultiples to a like Number of Magnitudes, each to the other, the ſame Multiple that one Magnitude is of one, ſo ſhall † all the Magnitudes be of all. And for the ſame Reaſon, L, and L, M, N, are * *Def.* 5 *of this.* † 1 *of this.*

† 5 *Def. of this.*

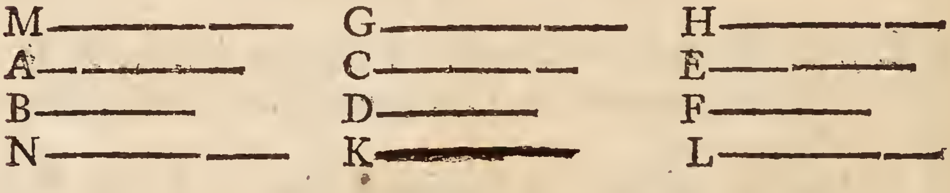
Equimultiples of B, and B, D, F. Therefore, as A is to B, ſo † is A, C, E, to B, D, F. Wherefore, *if there be any Number of Magnitudes proportional, as one of the Antecedents is to one of the Conſequents, ſo are all the Antecedents to all the Conſequents*; which was to be demonſtrated.

PROPOSITION XIII.

THEOREM.

If the firſt has the ſame Proportion to the ſecond, as the third to the fourth; and if the third has a greater Proportion to the fourth, than the fifth to the ſixth; then alſo ſhall the firſt have a greater Proportion to the ſecond, than the fifth has to the ſixth.

LET the firſt A have the ſame Proportion to the ſecond B, as the third C has to the fourth D; and let the third C have a greater Proportion to the fourth D, than the fifth E to the ſixth F. I ſay,



Likewiſe, that the firſt A to the ſecond B has a greater Proportion, than the fifth E to the ſixth F.

* 7 *Def. of this.*

For, becauſe C has a greater Proportion to D, than E has to F; there are * certain Equimultiples of C and E, and others of D and F, ſuch that the Multiple of C may exceed the Multiple of D; but the Multiple of E not that of F. Now let theſe Equimultiples of C and E, be G and H; and K and L, thoſe of D and F; ſo that G exceeds K, and H not L: Make M the ſame Multiple of A, as G is of C; and N the ſame of B, as K is of D.

† 5 *Def.*

Then, becauſe A is to B, as C is to D; and M and G are Equimultiples of A, C; and N, K, of B, D: If M exceeds N, then † G will exceed K; and if M be equal to N, G will be equal to K; and if leſs, leſs. But G does exceed K. Therefore M will alſo ſo exceed N. But H does not exceed L. And M, H, are Equimultiples of A, E; and N, L, any others of B, F. Therefore A has a * greater Proportion to B, than E has to F. Wherefore, *if the firſt has the ſame Proportion to the ſecond, as the third to the fourth; and if the third has a greater Proportion to the fourth, than the fifth to the ſixth; then alſo ſhall the firſt have a greater Proportion to the ſecond, than the fifth has to the ſixth*; which was to be demonſtrated.

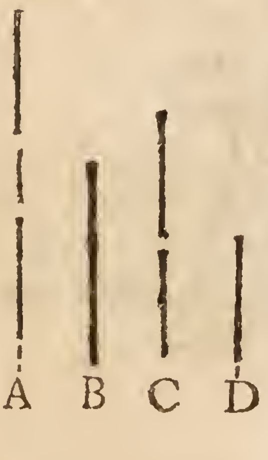
* 7 *Def. of this.*

PROPOSITION XIV.

THEOREM.

If the firſt has the ſame Proportion to the ſecond, as the third has to the fourth; and if the firſt be greater than the third; then will the ſecond be greater than the fourth. But if the firſt be equal to the third, then the ſecond ſhall be equal to the fourth; and if the firſt be leſs than the third, then the ſecond will be leſs than the fourth.

LET the firſt A have the ſame Proportion to the ſecond B, as the third C has to the fourth D: And let A be greater than C. I ſay, B is alſo greater than D.



For, becauſe A is greater than C, and B is any other Magnitude: A will have * greater Proportion to B, than C has to B; but as A is to B, ſo is C to D; therefore, alſo, C ſhall † have a greater Proportion to D, than C hath to B. But that Magnitude to which the ſame bears a greater Proportion, is ‡ the leſſer Magnitude: Wherefore D is leſs than B; and conſequently B will be greater than D. In like manner we demonſtrate, if A be equal to C, that B will be equal to D; and if A be leſs than C, that B will be leſs than D. Therefore, *if the firſt has the ſame Proportion to the ſecond, as the third has to the fourth; and if the firſt be greater than the third; then will the ſecond be greater than the*

* 8 *of this.*

† 13 *of this.*

‡ 10 *of this.*

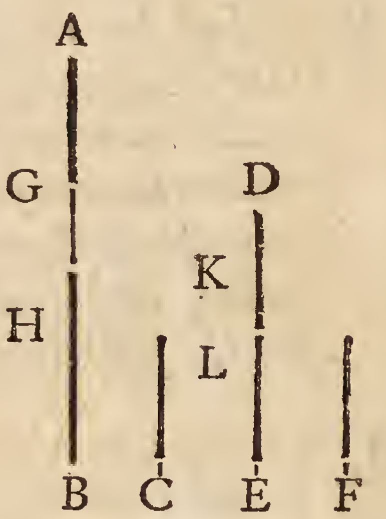
the fourth. But if the firſt be equal to the third, then the ſecond ſhall be equal to the fourth; and if the firſt be leſs than the third, then the ſecond will be leſs than the fourth; which was to be demonſtrated.

PROPOSITION XV.

THEOREM.

Parts have the ſame Proportion as their like Multiples, if taken correſpondently.

LET AB be the ſame Multiple of C, as DE is of F. I ſay, as C is to F, ſo is AB to DE.



For, becauſe AB and DE are Equimultiples of C and F, there ſhall be as many Magnitudes equal to C in AB, as there are Magnitudes equal to F in DE. Now, let AB be divided into the Magnitudes AG, GH, HB, each equal to C; and ED into the Magnitudes DK, KL, LE, each equal to F. Then the Number of the Magnitudes AG, GH, HB, will be equal to the Number of the Magnitudes DK, KL, LE. Now, becauſe AG, GH, HB, are equal, as likewiſe DK, KL, LE, it ſhall be * as AG is to DK, ſo is GH to KL, and ſo is HB to LE. But as one of the Antecedents is to one of the Conſequents, ſo † all the Antecedents to all the Conſequents. Therefore, as AG is to DK, ſo is AB to DE. But AG is equal to C, and DK to F. Whence, as C is to F, ſo ſhall AB be to DE. Therefore, *Parts have the ſame Proportion as their like Multiples, if taken correſpondently*; which was to be demonſtrated.

* *7 of this.*

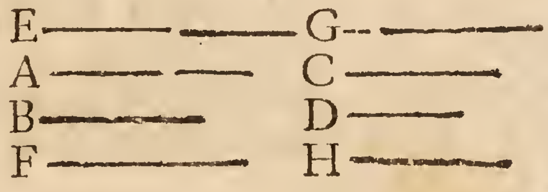
† *12 of this.*

PROPOSITION XVI.

THEOREM.

If four Magnitudes of the ſame Kind are proportional, they ſhall alſo be alternately proportional.

LET four Magnitudes A B C D, be proportional; whereof A is to B, as C is to D. I ſay likewiſe, that they will be alternately proportional, *viz.* as A is to C, ſo is B to D; for take E, F, Equimultiples of A and B; and G, H, any Equimultiples of C, D.



Then, becauſe E is the ſame Multiple of A, as F is of B, and Parts have the ſame Proportion * to their like Multiples, if taken correſpondently; it ſhall be as A is to B, ſo is E to F. But as A is to B, ſo is C to D. Therefore alſo, as C is to D, ſo † is E to F. Again, becauſe G, H, are Equimultiples of C and D, and Parts have the ſame Proportion with their like Multiples, if taken correſpondently, it will be as C is to D, ſo is G to H; but as C is to D, ſo is E to F. Therefore alſo, as E is to F, ſo is G to H; and if four Magnitudes be proportional, and the firſt greater than the third, then the ſecond will be ‡ greater than the fourth; and if the firſt be equal to the third, the ſecond will be equal to the fourth; and if leſs, leſs. Therefore, if E exceeds G, F will exceed H; and if E be equal to G, F will be equal to H; and if leſs, leſs. But E, F, are any Equimultiples of A, B; and G, H, any Equimultiples of C, D. Whence, as A is to C, ſo ſhall B be ‖ to D. Therefore, *if four Magnitudes of the ſame Kind are proportional, they ſhall alſo be alternately proportional.*

* 15 *of this.*

† 11 *of this.*

‡ 14 *of this.*

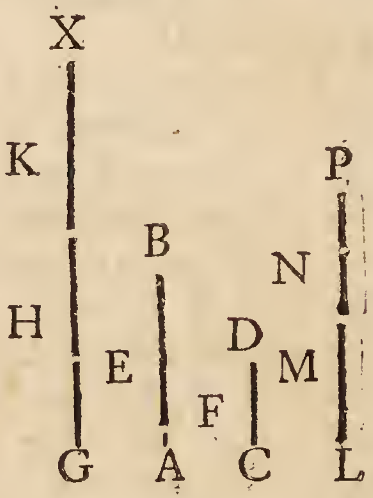
‖ *Def.* 5.

PROPOSITION XVII.

THEOREM.

If Magnitudes compounded are proportional, they ſhall alſo be proportional when divided.

LET the compounded Magnitudes AB, BE, CD, DF, be proportional; that is, let AB be to BE, as CD is to DF. I ſay, theſe Magnitudes divided are proportional, *viz.* as AE is to EB, ſo is CF to FD.



For let GH, HK, LM, MN, be Equimultiples of AE, EB, CF, FD; and KX, NP, any Equimultiples of EB, FD.

Because GH is the ſame Multiple of AE, as HK is of EB; therefore GH * is the ſame Multiple of AE, as GK is of AB. But GH is the ſame Multiple of AE, as LM is of CF. Wherefore GK is the ſame Multiple of AB, as LM is of CF. Again, becauſe LM is the ſame Multiple of CF, as MN is of FD, LM will be * the ſame Multiple of C F, as L~~M~~ is of CD. Therefore GK is the ſame Multiple of AB, as LN is of CD. And ſo GK, LN, will be Equimultiples of AB, CD. Again, becauſe HK is the ſame Multiple of EB, as MN is of FD; as likewiſe KX the ſame Multiple of EB, as NP is of FD, the compounded Magnitude HX is † alſo the ſame Multiple of EB, as MP is of FD. Wherefore, ſince it is as AB is to BE, ſo is CD to DF; and GK, LN, are Equimultiples of AB, CD; and alſo HX, MP, any Equimultiples of EB, FD: If GK exceeds HX, then LN will ‡ exceed MP: and if GK be equal to HX, then LN will be equal to MP; if leſs, leſs. Now let GK exceed HX; then if HK, which is common, be taken away, GH ſhall exceed KX. But when GK exceeds HX, then LN exceeds MP; therefore LN does exceed MP. If MN, which is common,

* 1 *of this.*

N

† 2 *of this.*

‡ *Def.* 5.

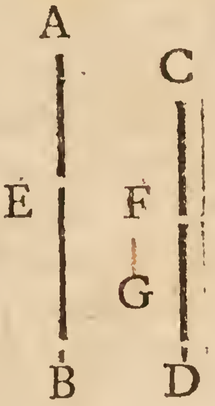
common, be taken away, then LM will exceed NP. And ſo, if GH exceeds KX, then LM will exceed NP. In like manner we demonſtrate, if GH be equal to KX, that LM will be equal to NP; and if leſs, leſs. But GH, LM, are Equimultiples of AE, CF; and KX, NP, are any Equimultiples of EB, FD. Whence, * as AE is to EB, ſo CF to FD. * *Def.* 5. Therefore, *if Magnitudes compounded are proportional, they ſhall alſo be proportional when divided;* which was to be demonſtrated.

PROPOSITION XVIII.

THEOREM.

If Magnitudes divided be proportional, the ſame alſo being compounded, ſhall be proportional.

LET the divided proportional Magnitudes AE, EB, CF, FD; that is, as AE is to EB, ſo is CF to FD. I ſay, they are alſo proportional when compounded; *viz.* as AB is to BE, ſo is CD to DF.



For if AB be not to BE, as CD is to DF, AB ſhall be to BE, as CD is to a Magnitude, either greater or leſs than FD.

Firſt, let it be to a leſſer, *viz.* to GD. Then, becauſe AB is to BE, as CD is to DG, compounded Magnitudes are proportional; and conſequently * they will be proportional when divided. * 17 *of this.* Therefore AE is to EB, as CG is to GD. But (by the *Hyp.*) as AE is to EB, ſo is CF to FD. Wherefore alſo, as CG is to GD, ſo † is CF to FD. But the firſt CG is greater † 11 *of this.* than the third CF; therefore the ſecond DG ſhall be ‡ greater than the fourth DF. But it is leſs, which is ‡ 14 *of this.* abſurd. Therefore AB is not to BE, as CD is to DG. We demonſtrate in the ſame manner, that AB to BE is not as CD to a greater than DF. Therefore AB to BE, muſt neceſſarily be as CD is to DF. And ſo, if Magnitudes divided be proportional, they will alſo be proportional when compounded; *which was to be demonſtrated.*

PRO-

PROPOSITION XIX.

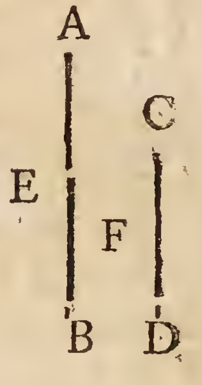
THEOREM.

If the Whole be to the Whole, as a Part taken away is to a Part taken away; then ſhall the Reſidue be to the Reſidue, as the Whole is to the Whole.

LET the Whole AB be to the Whole CD, as the Part taken away AE is to the Part taken away CF. I ſay, the Reſidue EB is to the Reſidue FD, as the Whole AB is to the Whole CD.

* 16 *of this.*
† 17 *of this.*

For, becauſe the Whole AB is to the Whole CD, as AE is to CF; it ſhall be * alternately as AB is to AE, ſo is CD to CF. Then, becauſe compounded Magnitudes, being proportional, will be † alſo proportional when divided; as BE is to EA, ſo is DF to FC: And again, it will be by Alternation, as BE to DF, ſo is EA to FC. But as EA to FC, ſo (by the *Hyp.*) is AB to CD. And therefore the Reſidue EB ſhall be to the Reſidue FD, as the Whole AB to the Whole CD. Wherefore, *if the Whole be to the Whole, as a Part taken away is to a Part taken away; then ſhall the Reſidue be to the Reſidue, as the Whole is to the Whole*; which was to be demonſtrated.



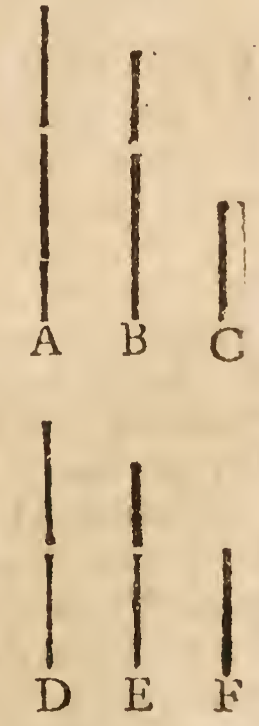
Coroll. If four Magnitudes be proportional, they will be likewiſe converſly proportional. For let AB be to BE, as CD to DF; then (by Alternation) it ſhall be as AB is to CD, ſo is BE to DF. Wherefore, ſince the Whole AB is to the Whole CD, as the Part taken away BE is to the Part taken away DF; the Reſidue AE to the Reſidue CF, ſhall be as the Whole AB to the Whole CD. And again, (by Inverſion and Alternation) as AB is to AE, ſo is CD to CF. Which is by converſe Ratio.

The Demonſtration of converſe Ratio, laid down in this Corollary, is only particular. For Alternation (which is uſed herein) cannot be applied but when the four proportional Magnitudes are all of the ſame Kind, as will appear from the 4th and 17th Definitions of this Book. But converſe Ratio may be uſed when the Terms *of the firſt Ratio are not of the ſame Kind with the Terms of the latter. Therefore, inſtead of that, it may not be improper to add this Demonſtration following: If four Magnitudes are proportional, they will be ſo converſly: For let* AB *be to* BE, *as* CD *to* DF. *And then dividing it is, as* AE *is to* BE, *ſo is* CF *to* DF: *And this inverſly is, as* BE *is to* AE, *ſo is* DF *to* CF; *which by compounding becomes, as* AB *is to* AE, *ſo is* CD *to* CF; *which by the* 17*th Definition is converſe Ratio: By* S. Cunn.

PROPOSITION XX.

THEOREM.

If there be three Magnitudes, and others equal to them in Number, which being taken two and two in each Order, are in the ſame Ratio; and if the firſt Magnitude be greater than the third, then the fourth will be greater than the ſixth: But if the firſt be equal to the third, then the fourth will be equal to the ſixth; and if the firſt be leſs than the third, the fourth will be leſs than the ſixth.



LET A, B, C, be three Magnitudes, and D, E, F, others equal to them in Number, taken two and two in each Order, are in the ſame Proportion; *viz.* let A be to B, as D is to E, and B to C, as E to F; and let the firſt Magnitude A be greater than the third C. I ſay, the fourth D is alſo greater than the ſixth F. And if A be equal to C, D is equal to F. But if A be leſs than C, D is leſs than F.

For, becauſe A is greater than C, and B is any other Magnitude; and ſince a greater Magnitude hath * a greater Proportion to the ſame Magnitude, than a leſſer hath, A will have a greater Proportion to B, than C to B. But as A is to B, ſo is D to E; and inverſly, as C is to B, ſo is F to E. Therefore alſo D will have a greater Proportion to E, than F has to E. But of Magnitudes having Proportion to the ſame Magnitude, that which has the greater

* 8 of this.

Proportion

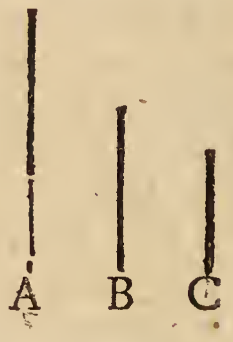
* 10 *of this.* Proportion is * the greater Magnitude. Therefore D is greater than F. In the ſame manner we demonſtrate, if A be equal to C, then D will be alſo equal to F; and if A be leſs than C, then D will be leſs than F. Therefore, *if there be three Magnitudes, and others equal to them in Number, which being taken two and two in each Order, are in the ſame Ratio; if the firſt Magnitude be greater than the third, then the fourth will be greater than the ſixth: But if the firſt be equal to the third, then the fourth will be equal to the ſixth; and if the firſt be leſs than the third, the fourth will be leſs than the ſixth;* which was to be demonſtrated.

PROPOSITION XXI.

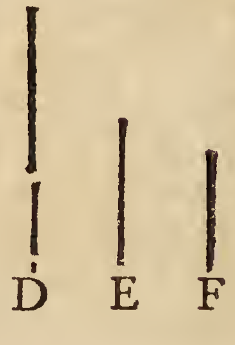
THEOREM.

If there be three Magnitudes, and others equal to them in Number, which taken two and two, are in the ſame Proportion, and the Proportion be perturbate; if the firſt Magnitude be greater than the third, then the fourth will be greater than the ſixth; but if the firſt be equal to the third, then is the fourth equal to the ſixth; if leſs, leſs.

LET three Magnitudes, A, B, C, be proportional; and others D, E, F, equal to them in Number. Let their Analogy likewiſe be perturbate, *viz.* as A is to B, ſo is E to F; and as B is to C, ſo is D to E; if the firſt Magnitude A be greater than the third C, I ſay, the fourth D is alſo greater than the ſixth F. And if A be equal to C, then D is equal to F; but if A be leſs than C, then D is leſs than F.



For ſince A is greater than C, and B is * 8 *of this.* ſome other Magnitude, A will have * a greater Proportion to B, than C has to B. But as A is to B, ſo is E to F; and inverſly, as C is to B, ſo is E to D. Wherefore alſo E ſhall have a greater Proportion to F, than E to D. But that Magnitude to which the ſame Magnitude has a greater Proportion, is † the leſſer Magnitude. † 10 *of this.*



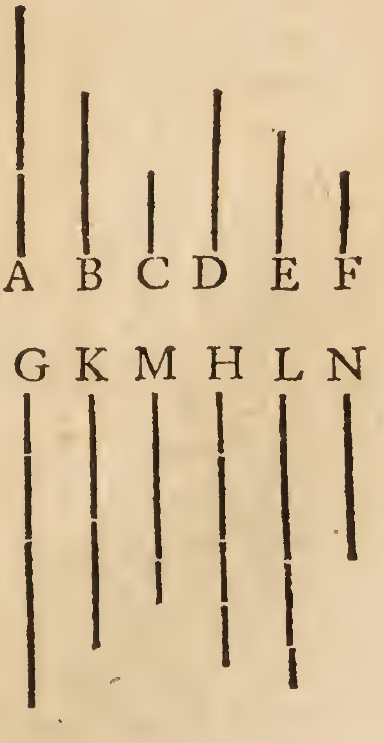
Therefore F is leſs than D; and ſo D ſhall be greater than F. After the ſame manner we demonſtrate, if A be equal to C, D will be alſo equal to F; and if A be leſs than C, D will alſo be leſs than F. *If, therefore, there are three Magnitudes, and others equal to them in Number, which taken two and two, are in the ſame Proportion, and the Proportion be perturbate; if the firſt Magnitude be greater than the third, then the fourth will be greater than the ſixth; but if the firſt be equal to the third, then is the fourth equal to the ſixth; if leſs, leſs;* which was to be demonſtrated.

PROPOSITION XXII.

THEOREM.

If there be any Number of Magnitudes, and others equal to them in Number, which taken two and two, are in the ſame Proportion; then they ſhall be in the ſame Proportion by Equality.

LET there be any Number of Magnitudes, A, B, C, and others, D, E, F, equal to them in Number, which taken two and two, are in the ſame Proportion, that is, as A is to B, ſo is D to E, and as B is to C, ſo is E to F. I ſay, they are alſo proportional by Equality, *viz.* as A is to C, ſo is D to F.



For let G, H, be Equimultiples of A, D; and K, L, any Equimultiples of B, E; and likewiſe M, N, any Equimultiples of C, F. Then, becauſe A is to B, as D is to E; and G, H, are Equimultiples of A, D; and K, L, Equimultiples of B, E; it ſhall be * as G is to K, ſo is H to L. For the ſame Reaſon alſo it will be, as K is to M, ſo is L to N. And ſince there are three Magnitudes G, K, M, and others H, L, N, equal to them in Number-

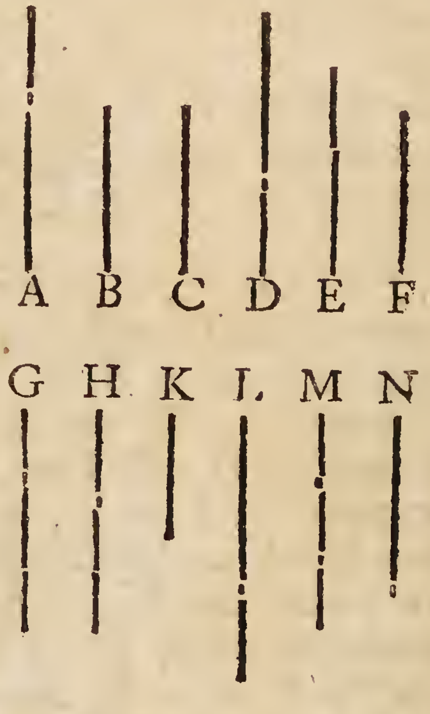
* 4 *of this.*

Number, which being taken two and two in each Order, are in the ſame Proportion; if G exceeds M, * 20 *of this.* *H will exceed N; if G be equal to M, then H ſhall be equal to N; and if G be leſs than M, H ſhall be leſs than N. But G, H, are Equimultiples of A, D; and M, N, any other Equimultiples of C and F. Whence, † *Def.* 5 *of this.* as A is to C, ſo ſhall † D be to F. Therefore, *if there be any Number of Magnitudes, and others equal to them in Number, which, taken two and two, are in the ſame Proportion; then they ſhall be in the ſame Proportion by Equality*; which was to be demonſtrated.

PROPOSITION XXIII

THEOREM.

If there be three Magnitudes, and others equal to them in Number, which, taken two and two, are in the ſame Proportion; and if their Analogy be perturbate, then ſhall they be alſo in the ſame Proportion by Equality.



LET there be three Magnitudes A, B, C, and others equal to them in Number, D, E, F, which, taken two and two, are in the ſame Proportion, and their Analogy be perturbate, that is, as A is to B, ſo is E to F; and as B is to C, ſo is D to E. I ſay, as A is to C, ſo is D to F.

For let G, H, L, be Equimultiples of A, B, D, and K, M, N, any Equimultiples of C, E, F.

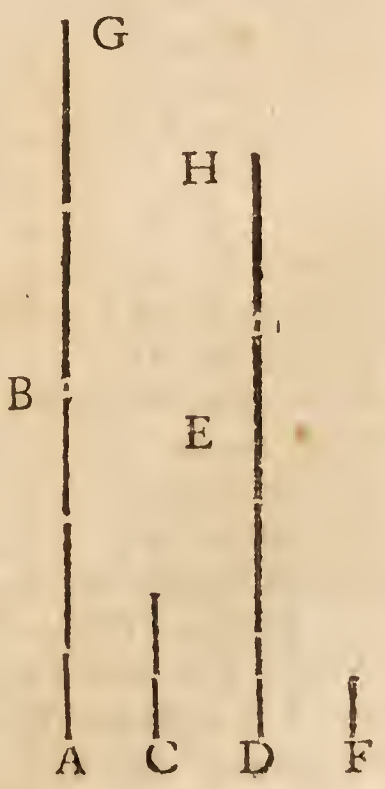
Then, becauſe G, H, are Equimultiples of A and B, and ſince Parts have the ſame Proportion as their like Multiples, when taken correſpondently, * 15 *of this.* it ſhall be * as A is to B, ſo is G to H; and by the ſame Reaſon, as E is to F, ſo is M to N. † 11 *of this.* But A is to B, as E is to F. Therefore, † as G is to H, ſo is M to N. Again, becauſe B is to C, as D is to E; and H, L, are Equimultiples of B and D; as likewiſe K, M, any Equimultiples of C, E; it ſhall be as H to K, ſo is L to M. But it has been alſo proved, that as G is to H, ſo is M to N. Therefore, becauſe three Magnitudes, G, H, K, and others, L, M, N, equal to them in Number, which taken two and two are in the ſame Proportion, and their Analogy is perturbate; then if G exceeds K, alſo L* will exceed N; and if G be equal to K, then L will be equal to N; and if G be leſs than K, L will likewiſe be leſs than N. But G, L, are Equimultiples of A, D; and K, N, Equimultiples of C, F. Therefore, as A is to C, ſo ſhall D be to F. Wherefore, *if there be three Magnitudes, and others equal to them in Number, which taken two and two are in the ſame Proportion; and if their Analogy be perturbate, then ſhall they be alſo in the ſame Proportion by Equality*; which was to be demonſtrated.

* *21 of this.*

PROPOSITION XXIV.

THEOREM.

If the firſt Magnitude has the ſame Proportion to the ſecond, as the third to the fourth; and if the fifth has the ſame Proportion to the ſecond, as the ſixth has to the fourth, then ſhall the firſt, compounded with the fifth, have the ſame Proportion to the ſecond, as the third, compounded with the ſixth, has to the fourth.



LET the firſt Magnitude AB have the ſame Proportion to the ſecond C, as the third DE has to the fourth F. Let alſo the fifth BG have the ſame Proportion to the ſecond C, as the ſixth EH has to the fourth F. I ſay, AG the firſt compounded with the fifth, has the ſame Proportion to the ſecond C, as DH the third compounded with the ſixth, has to the fourth F.

For, becauſe BG is to C, as EH is to F, it ſhall be
(inverſly) as C is to BG, ſo is F to EH. Then, ſince
AB is to C, as DE is to F; and as C is to BG, ſo is F
* 22 of this. to EH; it ſhall be * by Equality as AB is to BG, ſo is
DE to EH. And becauſe Magnitudes, being divided,
† 18 of this. are proportional, they ſhall alſo be † proportional when
compounded. Therefore, as AG is to GB, ſo is DH
‡ Hyp. to HE: But as GB is ‡ to C, ſo alſo is HE to F.
Wherefore, by Equality*, it ſhall be as AG is to C,
ſo is DH to F. Therefore, *if the firſt Magnitude has the ſame Proportion to the ſecond, as the third to the fourth; and if the fifth has the ſame Proportion to the ſecond, as the ſixth has to the fourth; then ſhall the firſt, compounded with the fifth, have the ſame Proportion to the ſecond, as the third, compounded with the ſixth, has to the fourth*; which was to be demonſtrated.

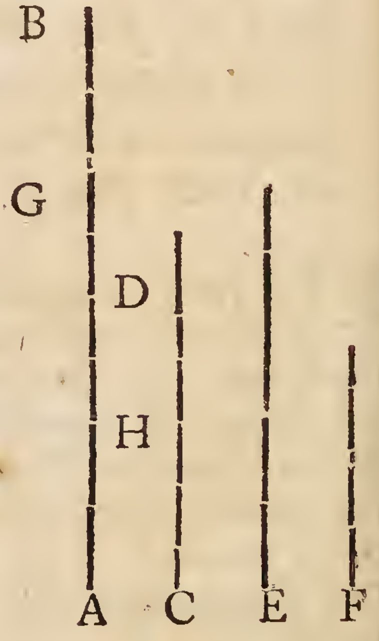
PROPOSITION XXV.

THEOREM.

If four Magnitudes be proportional, the greateſt and the leaſt of them will be greater than the other two.

LET four Magnitudes AB, CD, EF, be prooportional, whereof AB is to CD, as E is to F; let AB be the greateſt of them, and F the leaſt. I ſay, AB, and F, are greater than CD and E.

For let AG be equal to E, and CH to F. Then, becauſe AB is to CD, as E is to F; and ſince AG, and CH, are each equal to E and F, it ſhall be as AB is to DC, ſo is AG to CH. And becauſe the Whole AB is, to the Whole CD, as the Part taken away AG, is to the Part taken away CH; it ſhall
* 19 of this. alſo be * as the Reſidue GB to
the Reſidue HD, ſo is the Whole AB to the whole CD.
But AB is greater than CD; therefore alſo GB ſhall be greater than HD. And ſince AG is equal to E and



and CH to F, AG and F will be equal to CH and E. But if equal things are added to unequal things, the Wholes ſhall be unequal. Therefore GB, HD, being unequal, for GB is the greater: If AG, and F, are added to GB; and CH, and E, to HD; AB and F will neceſſarily be greater than CD and E. Wherefore, *if four Magnitudes be proportional, the greateſt and the leaſt of them will be greater than the other two*; which was to be demonſtrated.

The END *of the* FIFTH BOOK.

EUCLID's

ELEMENTS.

BOOK VI.

DEFINITIONS.

I. *SIMILAR Right-lined Figures are ſuch as have each of their ſeveral Angles equal to one another, and the Sides about the equal Angles proportional to each other.*

II. *Figures are ſaid to be reciprocal, when the antecedent and conſequent Terms of the Ratio's are in each Figure.*

III. *A Right Line is ſaid to be cut into mean and extreme Proportion, when the Whole is to the greater Segment, as the greater Segment is to the leſſer.*

IV. *The Altitude of any Figure is a perpendicular Line drawn from the Top, or Vertex, to the Baſe.*

V. *A Ratio is ſaid to be compounded of Ratio's, when the Quantities of the Ratio's, being multiplied into one another, do produce a Ratio.*

PROPOSITION I.

THEOREM.

Triangles and Parallelograms, that have the ſame Altitude, are to each other as their Baſes.

LET the Triangles ABC, ACD, and the Parallelograms, EC, CF, have the ſame Altitude, *viz.* the Perpendicular drawn from the Point A to BD. I ſay, as the Baſe BC is to the Baſe CD, ſo is the Triangle ABC, to the Triangle ACD; and ſo is the Parallelogram EC to the Parallelogram CF.

For, produce BD both ways to the Points H and L; and take GB, GH, any Number of Times equal to the Baſe BC; and DK, KL, any Number of Times equal to the Baſe CD; and join AG, AH, AK, AL.

Then, becauſe CB, BG, GH, are equal to one another, the Triangles AHG, AGB, ABC, alſo will be * equal to one another: Therefore the ſame Multiple that the Baſe HC is of BC, ſhall the Triangle AHC be of the Triangle ABC. By the ſame Reaſon, the ſame Multiple that the Baſe LC is of the Baſe CD, ſhall the Triangle ALC be of the Triangle ACD. And if HC be equal to the Baſe CL, the Triangle AHC is alſo * equal to the Triangle ALC: And if the Baſe HC exceeds the Baſe CL, then the Triangle AHC will exceed the Triangle ALC. And if HC be leſs, then the Triangle AHC will be leſs. Therefore ſince there are four Magnitudes, *viz.* the two Baſes BC, CD, and the two Triangles ABC, ACD; and ſince the Baſe HC, and the Triangle AHC, are Equimultiples of the Baſe BC, and the Triangle ABC: And the Baſe CL, and the Triangle ALC, are Equimultiples of the Baſe CD, and the Triangle ADC: And it has been proved, that if the Baſe HC exceeds the Baſe CL, the Triangle AHC will exceed the Triangle ALC; and if equal, equal; if leſs, leſs: Then, as the Baſe BC is to the Baſe CD, ſo † is the Triangle ABC, to the Triangle ACD.

* 38. 1.

† *Def.* 5. 5.

And

† 41. 1. And becauſe the Parallelogram EC is † double to the Triangle ABC; and the Parallelogram FC double † to the Triangle ACD; and Parts have the ſame Proportion as their like Multiples: Therefore as the Triangle ABC is to the Triangle ACD, ſo is the Parallelogram EC to the Parallelogram CF. And ſo, ſince it has been proved, that the Baſe BC is to the Baſe CD, as the Triangle ABC is to the Triangle ACD; and the Triangle ABC is to the Triangle ACD, as the Parallelogram EC is to the Parallelogram CF; it ſhall be ‡ as the Baſe BC is to the Baſe CD, ſo is the Parallelogram EC to the Parallelogram FC. Wherefore *Triangles and Parallelograms, that have the ſame Altitude, are to each other as their Baſes*; which was to be demonſtrated. † 11. 5.

PROPOSITION II.

THEOREM.

If a Right Line be drawn parallel to one of the Sides of a Triangle, it ſhall cut the Sides of the Triangle proportionally; and if the Sides of the Triangle be cut proportionally, then a Right Line joining the Points of Section, ſhall be parallel to the other Side of the Triangle.

LET DE be drawn parallel to BC, a Side of the Triangle ABC. I ſay, DB is to DA, as CE is to EA.

For, let BE, CD, be joined.

† 37. 1. Then the Triangle BDE is * equal to the Triangle CDE; for they ſtand upon the ſame Baſe DE, and are between the ſame Parallels DE and BC; and ADE is ſome other Triangle. But equal Magnitudes have † the ſame Proportion to one and the ſame Magnitude. Therefore, as the Triangle BDE is to the Triangle ADE, ſo is the Triangle CDE to the Triangle ADE. † 7. 5.

‡ 1 *of this.* But as the Triangle BDE is to the Triangle ADE, ſo ‡ is BD to DA; for ſince they have the ſame Altitude, *viz.* a Perpendicular drawn from the Point E to AB, they are to each other as their Baſes. And for the ſame Reaſon, as the Triangle CDE is

to the Triangle ADE, ſo is CE to EA: And therefore, as BD is to DA, ſo * is CE to EA. * 11. 5.

And if the Sides AB, AC, of the Triangle ABC, be cut proportionally, that is, ſo that BD be to DA, as CE is to EA; and if DE be joined, I ſay, DE is parallel to BC.

For the ſame Conſtruction remaining, becauſe BD is to DA, as CE is to EA; and BD is † to DA, as † 1 *of this.* the Triangle BDE is to the Triangle ADE; and CE is to EA, as the Triangle CDE is to the Triangle ADE: It ſhall be as the Triangle BDE is to the Triangle ADE, ſo is * the Triangle CDE to the Triangle ADE. And ſince the Triangles BDE, CDE, have the ſame Proportion to the Triangle ADE, the Triangle BDE ſhall be † equal to the † 9. 5. Triangle CDE; and they have the ſame Baſe DE: But equal Triangles, being upon the ſame Baſe, ‡ are ‡ 39. 1. between the ſame Parallels; therefore DE is parallel to BC. Wherefore, *if a Right Line be drawn parallel to one of the Sides of a Triangle, it ſhall cut the Sides of the Triangle proportionally; and if the Sides of the Triangle be cut proportionally, then a Right Line, joining the Points of Section, ſhall be parallel to the other Side of the Triangle*; which was to be demonſtrated.

PROPOSITION III.

Theorem.

If one Angle of a Triangle be biſected, and the Right Line that biſects the Angle, cuts the Baſe alſo; then the Segments of the Baſe will have the ſame Proportion, as the other Sides of the Triangle. And if the Segments of the Baſe have the ſame Proportion that the other Sides of the Triangle have; then a Right Line drawn from the Vertex, to the Point of Section of the Baſe, will biſect the Angle of the Triangle.

LET there be a Triangle ABC, and let its Angle BAC be * biſected by the Right Line AD. I * 9. 1. ſay, as BD is to DC, ſo is BA to AC.

* 31. 1. For, thro' C draw * CE parallel to DA, and pro-
duce BA till it meets CE in the Point E.

Then, becauſe the Right Line AC falls on the
† 29. 1. Parallels AD, EC, the Angle ACE will be † equal
to the Angle CAD: But the Angle CAD (by the
Hypotheſis) is equal to the Angle BAD. Therefore
the Angle BAD will be equal to the Angle ACE.
Again, becauſe the Right Line BAE falls on the
Parallels AD, EC, the outward Angle BAD is
† equal to the inward Angle AEC; but the Angle
ACE has been proved equal to the Angle BAD:
Therefore ACE ſhall be equal to AEC; and ſo the
‡ 6. 1. Side AE is equal ‡ to the Side AC. And becauſe the
Line AD is drawn parallel to CE, the Side of the
* 2 *of this.* Triangle BCE, it ſhall be * as BD is to DC, ſo is
BA to AE; but AE is equal to AC. Therefore, as
† 7. 5. BD is to DC, ſo is † BA to AC.

And if BD be to DC, as BA is to AC; and the
Right Line AD be joined, then, I ſay, the Angle
BAC is biſected by the Right Line AD.

For the ſame Conſtruction remaining, becauſe BD
is to DC, as BA is to AC; and as BD is to DC, ſo
‡ 2 *of this.* is ‡ BA to AE; for AD is drawn parallel to one
Side EC of the Triangle BCE; it ſhall be as BA is
to AC, ſo is BA to AE. Therefore AC is equal
to AE; and accordingly the Angle AEC is equal
to the Angle ECA: But the Angle AEC is equal
* 25. 1. * to the outward Angle BAD; and the Angle ACE,
equal * to the alternate Angle CAD. Wherefore
the Angle BAD is alſo equal to the Angle CAD;
and ſo the Angle BAC is biſected by the Right Line
AD. Therefore, *if the Angle of a Triangle be biſected, and the Right Line that biſects the Angle, cuts the Baſe alſo; then the Segments of the Baſe will have the ſame Proportion, as the other Sides of the Triangle. And if the Segments of the Baſe have the ſame Proportion that the other Sides of the Triangle have; then a Right Line drawn from the Vertex, to the Point of Section of the Baſe, will biſect the Angle of the Triangle*; which was to be demonſtrated.

PROPOSITION IV.

THEOREM.

The Sides about the equal Angles of equiangular Triangles, are proportional; and the Sides which are ſubtended under the equal Angles, are homologous, or of like Ratio.

LET ABC, DCE, be equiangular Triangles, having the Angle ABC equal to the Angle DCE, the Angle ACB equal to the Angle DEC, and the Angle BAC equal to the Angle CDE. I ſay, the Sides that are about the equal Angles of the Triangles ABC, DCE, are proportional; and the Sides that are ſubtended under the equal Angles, are homologous, or of like Ratio.

Set the Side BC in the ſame Right Line with the
Side CE; and becauſe the Angles ABC, ACB, are
* leſs than two Right Angles, and the Angle ACB * 17. 1.
is equal to the Angle DEC, the Angles ABC, DEC,
are leſs than two Right Angles. And ſo BA, ED,
produced, will meet † each other; let them be produced, † *Ax.* 12.
and meet in the Point F. Then, becauſe the
Angle DCE is equal to the Angle ABC, BF ſhall
be ‡ parallel to DC. Again, becauſe the Angle ACB ‡ 28. 1.
is equal to the Angle DEC, the Side AC will be ‡
parallel to the Side FE; therefore FACD is a Parallelogram;
and conſequently FA is * equal to DC, * 34. 1.
and AC to FD; and becauſe AC is drawn parallel
to FE, the Side of the Triangle FBE, it ſhall † be † 2 *of this.*
as BA is to AF, ſo is BC to CE; and (by Alternation)
as BA is to BC, ſo is CD to CE. Again;
becauſe CD is parallel to BF, it ſhall be † as BC is
to CE, ſo is FD to DE; but FD is equal to AC.
Therefore as BC is to CE, ſo is ‡ AC to DE: And ‡ 7. 5.
ſo (by Alternation) as BC is to CA, ſo is CE to ED.
Wherefore, becauſe it is demonſtrated, that AB is to
BC, as DC is to CE; and as BC is to CA, ſo is
CE to ED; it ſhall be * by Equality, as BA is to * 21. 5.
AC, ſo is CD to DE. Therefore *the Sides about
the equal Angles of equiangular Triangles, are proportional; and the Sides which are ſubtended under the*

equal Angles, are homologous, or of like Ratio; which was to be demonſtrated.

PROPOSITION V.

THEOREM.

If the Sides of two Triangles are proportional, the Triangles ſhall be equiangular; and their Angles, under which the homologous Sides are ſubtended, are equal.

LET there be two Triangles, ABC, DEF, having their Sides proportional, that is, let AB be to BC, as DE is to EF; and as BC to CA, ſo is EF to FD. And alſo as BA to CA, ſo ED to DF. I ſay, the Triangle ABC is equiangular to the Triangle DEF; and the Angles equal, under which the homologous Sides are ſubtended, *viz.* the Angle ABC equal to the Angle DEF; and the Angle BCA equal to the Angle EFD; and the Angle BAC equal to the Angle EDF.

For, at the Points E and F, with the Line EF, make * the Angle FEG, equal to the Angle ABC; and the Angle EFG, equal to the Angle BCA: Then the remaining Angle BAC is † equal to the remaining Angle EGF.

* 23. 1.

† *Cor.* 32. 1.

And ſo the Triangle ABC is equiangular to the Triangle EGF; and conſequently the Sides that are ſubtended under the equal Angles, are proportional. Therefore, as AB is to BC, ſo is ‡ GE to EF; but as AB is to BC, ſo is DE to EF: Therefore, as DE is to EF, ſo is * GE to EF. And ſince DE, EG, have the ſame Proportion to EF, DE ſhall be † equal to EG. For the ſame Reaſon, DF is equal to FG; but EF is common. Then, becauſe the two Sides DE, EF, are equal to the two Sides GE, EF, and the Baſe DF is equal to the Baſe FG, the Angle DEF is ‡ equal to the Angle GEF; and the Triangle DEF equal to the Triangle GEF; and the other Angles of the one, equal to the other Angles of the other, which are ſubtended by the equal Sides. Therefore the Angle DEF is equal to the Angle GEF, and the Angle EDF equal to the Angle EGF.

‡ 4 *of this.*

* 11. 5.

† 9. 5.

‡ 8. 1.

EGF. And becauſe the Angle DEF is equal to the Angle GEF; and the Angle GEF equal to the Angle ABC; therefore the Angle ABC ſhall be alſo equal to the Angle FED: For the ſame Reaſon, the Angle ACB ſhall be equal to the Angle DFE; as alſo the Angle A equal to the Angle D; therefore the Triangle ABC will be equiangular to the Triangle DEF. Wherefore, *if the Sides of two Triangles are proportional, the Triangles ſhall be equiangular; and their Angles, under which the homologous Sides are ſubtended, are equal*; which was to be demonſtrated.

PROPOSITION VI.

THEOREM.

If two Triangles have one Angle of the one, equal to one Angle of the other; and if the Sides about the equal Angles be proportional, then the Triangles are equiangular, and have thoſe Angles equal, under which are ſubtended the homologous Sides.

LET there be two Triangles ABC, DEF, having one Angle BAC of the one equal to the Angle EDF of the other; and let the Sides about the equal Angles be proportional, *viz.* let AB be to AC, as ED is to DF. I ſay, the Triangle ABC is equiangular to the Triangle DEF; and the Angle ABC equal to the Angle DEF; and the Angle ACB equal to the Angle DFE.

For at the Points D and F, with the Right Line DF, make * the Angle FDG equal to either of the Angles BAC, EDF; and the Angle DFG equal to the Angle ACB. * 23. 1.

Then the other Angle B is † equal to the other Angle G; and ſo the Triangle ABC is equiangular to the Triangle DGF; and conſequently, as BA is to AC, ſo is ‡ GD to DF: But (by the *Hyp.*) as BA is to AC, ſo is ED to DF. Therefore, as ED is * to DF, ſo is GD to DF; whence ED is † equal to DG, and DF is common; therefore the two Sides ED, DF, are equal to the two Sides GD, DF; and the Angle EDF, equal to the Angle GDF: Con-

† Cor. 32. 1.
‡ 4 *of this.*
* 11. 5.
† 9. 5.

* 4. 1. Conſequently the Baſe E F is * equal to the Baſe F G, and the Triangle D E F equal to the Triangle G D F, and the other Angles of the one equal to the other Angles of the other, each to each; under which the equal Sides are ſubtended. Therefore the Angle D F G is equal to the Angle D F E, and the Angle G, equal to the Angle E; but the Angle D F G is equal to the Angle A C B: Wherefore the Angle A C B is equal to the Angle D F E; and the Angle B A C is ‡ alſo equal to the Angle E D F: Therefore the other Angle at B is equal to the other Angle at E; and ſo the Triangle A B C is equiangular to the Triangle D E F. Therefore, *if two Triangles have one Angle of the one, equal to one Angle of the other; and if the Sides about the equal Angles be proportional, then the Triangles are equiangular, and have thoſe Angles equal, under which are ſubtended the homologous Sides*; which was to be demonſtrated.

‡ *By Hyp.*

PROPOSITION VII.

THEOREM.

If there are two Triangles, having one Angle of the one equal to one Angle of the other, and the Sides about other Angles proportional; and if the remaining third Angles are either both leſs, or both not leſs, than Right Angles, then ſhall the Triangles be equiangular, and have thoſe Angles equal, about which are the proportional Sides.

LET two Triangles A B C, D E F, have one Angle of the one equal to one Angle of the other, *viz.* the Angle B A C equal to the Angle E D F; and let the Sides about the other Angles A B C, D E F, be proportional; *viz.* as D E is to E F, ſo let A B be to B C; and let the other Angles at C and F, be both leſs, or both not leſs, than Right Angles. I ſay, the Triangle A B C is equiangular to the Triangle D E F; and the Angle A B C is equal to the Angle D E F; as alſo the other Angle at C equal to the other Angle at F.

For, if the Angle A B C be not equal to the Angle D E F, one of them will be the greater, which let be A B C.

ABC. Then at the Point B, with the Right Line AB, make * the Angle ABG equal to the Angle DEF. *23. 1.

Now, becauſe the Angle A is equal to the Angle D, and the Angle ABG, equal to the Angle DEF; the remaining Angle AGB is † equal to the remaining Angle DFE: And therefore the Triangle ABG is equiangular to the Triangle DEF; and ſo, as AB is to BG, ſo is ‡ DE to EF; but as DE is to EF, ſo is * AB to BC. Therefore as AB is to BC, ſo is AB to BG; and ſince AB has the ſame Proportion to BC that it has to BG, BC ſhall be † equal to BG; and conſequently the Angle at C equal to the Angle BGC. Wherefore each of the Angles BCG, or BGC, is leſs than a Right Angle; and conſequently, AGB is greater than a Right Angle. But the Angle AGB has been proved equal to the Angle at F; therefore the Angle at F, is greater than a Right Angle: But (by the *Hyp.*) it is not greater, ſince C is not greater than a Right Angle, which is abſurd. Wherefore the Angle ABC is not unequal to the Angle DEF; and ſo it muſt be equal to the ſame; but the Angle at A is equal to that at D; wherefore the Angle remaining at C is equal to the remaining Angle at F; and conſequently the Triangle ABC is equiangular to the Triangle DEF. Therefore, *if there are two Triangles having one Angle of the one equal to one Angle of the other, and the Sides about other Angles proportional; and if the remaining third Angles are either both leſs, or both not leſs, than Right Angles, then ſhall the Triangles be equiangular, and have thoſe Angles equal, about which are the proportional Sides*; which was to be demonſtrated.

† *Cor.* 32. 1.
‡ 4 *of this.*
* *By Hyp.*
† 9. 5.

PRO-

PROPOSITION VIII.

THEOREM.

If a Perpendicular be drawn, in a Right-~~lined~~ Angle'd Triangle, from the Right Angle to the Baſe, then the Triangles on each Side of the Perpendicular are ſimilar both to the Whole, and alſo to one another.

LET ABC be a Right-angled Triangle, whoſe Right Angle is BAC; and let the Perpendicular AD be drawn from the Point A to the Baſe BC. I ſay, the Triangles ABD, ADC, are ſimilar to one another, and to the whole Triangle ABC.

For, becauſe the Angle BAC is equal to the Angle ADB, for each of them is a Right Angle; and the Angle at B is common to the two Triangles ABC, ABD; the remaining Angle ACB ſhall be * equal to the remaining Angle BAD. Therefore the Triangle ABC is equiangular to the Triangle ABD; and ſo as † BC, which ſubtends the Right Angle of the Triangle ABC, is to BA, ſubtending the Right Angle of the Triangle ABD, ſo is AB, ſubtending the Angle C of the Triangle ABC, to DB, ſubtending an Angle equal to the Angle C, *viz.* the Angle BAD, of the Triangle ABD. And ſo moreover is AC to AD, ſubtending the Angle B, which is common to the two Triangles. Therefore the Triangle ABC is ‡ equiangular to the Triangle ABD; and the Sides about the equal Angles are proportional. Wherefore the Triangle ABC is ‡ ſimilar to the Triangle ABD. By the ſame way we demonſtrate, that the Triangle ADC is alſo ſimilar to the Triangle ABC. Wherefore each of the Triangles ABD, ADC, is ſimilar to the whole Triangle.

* *Cor.* 32. 1.

† 4 *of this.*

‡ *Def.* 1 *of this.*

I ſay, the ſaid Triangles are alſo ſimilar to one another.

For, becauſe the Right Angle BDA is equal to the Right Angle ADC, and the Angle BAD has been proved equal to the Angle C; it follows, that the remaining Angle at B ſhall be equal to the remaining Angle DAC. And ſo the Triangle ABD is equiangular to the Triangle ADC. Wherefore, as † BD, ſubtending the Angle BAD of the Triangle ABD, is to DA, ſubtending the Angle at C of the Triangle ADC, which is equal to the Angle BAD; ſo is AD, ſubtending the Angle B of the Triangle ABD, to DC, ſubtending the Angle DAC, equal to the Angle B. And moreover, ſo is BA to AC, ſubtending the Right Angles at D; and conſequently the Triangle ABD is ſimilar to the Triangle ADC. Wherefore, *if a Perpendicular be drawn, in a Right-angled Triangle, from the Right Angle to the Baſe, then the Triangles on each Side of the Perpendicular are ſimilar both to the Whole, and alſo to one another*; which was to be demonſtrated. † 4 *of this.*

Coroll. From hence it is manifeſt, that the Perpendicular drawn in a Right-angled Triangle from the Right Angle to the Baſe, is a mean Proportional between the Segments of the Baſe. Moreover, either of the Sides containing the Right Angle, is a mean Proportional between the whole Baſe, and that Segment thereof which is next to the Side.

viz. BD : AD :: AD : DC
and BC : BA :: BA : BD
so BC : AC :: AC : DC

PROPOSITION IX.

PROBLEM.

To cut off any Part required from a given Right Line.

LET AB be a Right Line given; from which muſt be cut off any required Part; ſuppoſe a third.

Draw any Right Line AC from the Point A, making an Angle at Pleaſure with the Line AB. Aſſume any Part D in the Line AC; make * DE, EC, each equal to AD; join BC; and draw † DF thro' D, parallel to BC. * 3. 1. † 31. 1.

Then, becauſe FD is drawn parallel to the Side BC of the Triangle ABC, it ſhall be ‡ as CD is to DA, ſo is BF to FA. But CD is double to DA. Therefore BF ſhall be double to FA; and ſo BA is triple to AF. Wherefore there is cut off AF, a third Part required of the given Right Line AB; *which was to be done.* ‡ 2 *of this.*

PRO-

PROPOSITION X.

PROBLEM.

To divide a given undivided Right Line, as another given Right Line is divided.

LET AB be a given undivided Right Line, and AC a divided Line. It is required to divide AB, as AC is divided.

Let AC be divided in the Points D and E, and ſo placed, as to contain any Angle with AB. Join the Points C and B; thro' D and E let DF, EG, be drawn * parallel to BC; and through D draw DHK, parallel to AB. * 31. 1.

Then FH, HB, are each of them Parallelograms; and ſo DH is † equal to FG, and HK to GB. And becauſe HE is drawn parallel to the Side KC, of the Triangle DKC, it ſhall be ‡ as CE is to ED, ſo is KH to HD. But KH is equal to BG, and HD to GF. Therefore, as CE is to ED, ſo is BG to GF. Again, becauſe FD is drawn parallel to the Side EG, of the Triangle AGE, as ED is to DA, ſo ſhall ‡ GF be to FA. But it has been proved, that CE is to ED as BG is to GF. Therefore, as CE is to ED, ſo is BG to GF; and as ED is to DA, ſo is GF to FA. Wherefore the given undivided Line AB is divided as the given Line AC is; *which was to be done.* † 34. 1. ‡ 2 *of this.*

PROPOSITION XI.

PROBLEM.

Two Right Lines being given, to find a third Proportional to them.

LET AB, AC, be two given Right Lines, ſo placed, as to make any Angle with each other. It is required to find a third Proportional to AB, AC.

Produce AB, AC, to the Points D and E; make BD equal to AC; join the Points B, C; and draw * the Right Line DE thro' D parallel to BC. * 31. 1.

Then, becauſe BC is drawn parallel to the Side
DE, of the Triangle ADE, it ſhall be * as AB is to * 2 *of this.*
BD, ſo is AC to CE. But BD is equal to AC.
Hence, as AB is to AC, ſo is AC to CE. Therefore a third Proportional CE is found to two given Right Lines AB, AC; *which was to be done.*

PROPOSITION XII.

PROBLEM.

Three Right Lines being given, to find a fourth Proportional to them.

LET A, B, C, be three Right Lines given. It is required to find a fourth Proportional to them.

Let DE and DF be two Right Lines, making any
Angle EDF with each other. Now make DG equal
to A, GE equal to B, DH equal to C; and draw the
Line GH, as alſo † EF thro' E, parallel to GH. † 31. 1.

Then, becauſe GH is drawn parallel to EF, the Side of the Triangle DEF, it ſhall be as DG is to GE, ſo is DH to HF. But DG is equal to A, GE to B, and DH to C. Conſequently, as A is to B, ſo is C to HF. Therefore the Right Line HF, a fourth Proportional to the three given Right Lines A, B, C, is found; *which was to be done.*

PROPOSITION XIII.

PROBLEM.

To find a mean Proportional between two given Right Lines.

LET the two given Right Lines be AB, BC. It
is required to find a mean Proportional between
them. Place AB, BC, in a direct Line, and on the
Whole AC deſcribe the Semicircle ADC, and * draw * 11. 1.
BD at Right Angles to AC from the Point B, and
let AD, DC, be joined.

Then, becauſe the Angle ADC, in a Semicircle,
is † a Right Angle; and ſince the Perpendicular DB is † 31. 3.
drawn from the Right Angle to the Baſe; therefore

DB

* Cor. 8. *of this.* DB is * a mean Proportional between the Segments of the Baſe AB, BC. Wherefore, a mean Proportional between the two given Lines AB, BC, is found; *which was to be done.*

PROPOSITION XIV.

THEOREM.

Equal Parallelograms, having one Angle of the one equal to one Angle of the other, have the Sides about the equal Angles reciprocal; and thoſe Parallelograms that have one Angle of the one equal to one Angle of the other, and the Sides that are about the equal Angles reciprocal, are equal between themſelves.

LET AB, BC, be equal Parallelograms, having the Angles at B equal; and let the Sides DB, BE, be in one ſtrait Line; then alſo will * the Sides FB, BG, be in one ſtrait Line. I ſay, the Sides of the Parallelograms AB, BC, that are about the equal Angles, are reciprocal; that is, as DB is to BE, ſo is GB to BF. * 14. 1.

For, let the Parallelogram FE be completed.

Then, becauſe the Parallelogram AB is equal to the Parallelogram BC, and FE is ſome other Parallelogram; it ſhall be as AB is to FE, ſo is † BC to FE; but as AB is to FE, ſo is ‡ DB to BE; and as BC is to FE, ſo is GB to BF. Therefore, as DB is to BE, ſo is GB to BF. Wherefore the Sides of the Parallelograms AB, BC, that are about the equal Angles, are reciprocally proportional. † 7. 5. ‡ 1 *of this.*

And if the Sides that are about the equal Angles, are reciprocally proportional, *viz.* if BD be to BE, as GB is to BF; I ſay, the Parallelogram AB is equal to the Parallelogram BC.

For, ſince DB is to BE, as GB is to BF; and DB to BE, as the Parallelogram AB ‡ to the Parallelogram FE; and GB ‡ to BF, as the Parallelogram BC to the Parallelogram FE; it ſhall be as AB is to FE, ſo is BC to FE. Therefore the Parallelogram AB is equal to the Parallelogram BC. And ſo *equal Parallelograms, having one Angle of the one equal to one Angle of the other, have the Sides about the equal Angles*

gles reciprocal; *and thoſe Parallelograms that have one Angle of the one equal to one Angle of the other, and the Sides that are about the equal Angle reciprocal, are equal between themſelves*; which was to be demonſtrated.

PROPOSITION XV.

THEOREM.

Equal Triangles, having one Angle of the one equal to one Angle of the other, have their Sides about the equal Angles reciprocal; *and thoſe Triangles that have one Angle of the one equal to one Angle of the other, and have alſo the Sides about the equal Angles reciprocal, are equal between themſelves.*

LET the equal Triangles ABC, ADE, have one Angle of the one equal to one Angle of the other, *viz.* the Angle BAC equal to the Angle DAE. I ſay, the Sides about the equal Angles are reciprocal; that is, as CA is to AD, ſo is EA to AB.

For, place CA and AD in one ſtrait Line, then EA and AB ſhall be* alſo in one ſtrait Line, and let BD be joined. Then, becauſe the Triangle ABC is equal to the Triangle ADE, and ABD is ſome other Triangle, the Triangle CAB ſhall be † to the Triangle BAD, as the Triangle ADE is to the Triangle BAD. But, as the Triangle CAB is to the Triangle BAD, ſo is CA ‡ to AD; and as the Triangle EAD is to the Triangle BAD, ſo ‡ is EA to AB. Therefore, as CA is to AD, ſo is EA to AB. Wherefore the Sides of the Triangles ABC, ADE, about the equal Angles, are reciprocal.

* 14. 1.
† 7. 5.
‡ 1 *of this.*

And, if the Sides about the equal Angles of the Triangles ABC, ADE, be reciprocal, *viz.* if CA be to AD as EA is to AB; I ſay, the Triangle ABC is equal to the Triangle ADE.

For, again let BD be joined. Then, becauſe CA is to AD, as EA is to AB; and CA to AD‡, as the Triangle ABC to the Triangle BAD; and EA to AB‡, as the Triangle EAD to the Triangle BAD; therefore, as the Triangle ABC is to the Triangle BAD, ſo ſhall the Triangle EAD be to the Triangle

BAD. Whence the Triangles ABC, ADE, have the ſame Proportion to the Triangle BAD: And ſo the Triangle ABC is equal to the Triangle ADE. Therefore *equal Triangles, having one Angle of the one equal to one Angle of the other, have their Sides about the equal Angles reciprocal; and thoſe Triangles that have one Angle of the one equal to one Angle of the other, and have alſo the Sides about the equal Angles reciprocal, are equal between themſelves*; which was to be demonſtrated.

PROPOSITION XVI.

THEOREM.

If four Right Lines be proportional, the Rectangle contained under the Extremes is equal to the Rectangle contained under the Means; and if the Rectangle contained under the Extremes be equal to the Rectangle contained under the Means, then are the four Right Lines proportional.

LET four Right Lines AB, CD, E, F, be proportional, ſo that AB be to CD, as E is to F. I ſay, the Rectangle contained under the Right Lines AB and F, is equal to the Rectangle contained under the Right Lines CD and E.

For, draw AG, CH, from the Points A, C, at Right Angles to AB and CD; and make AG equal to F, and CH equal to E; and let the Parallelograms BG, DH, be completed.

Then, becauſe AB is to CD, as E is to F, and ſince CH is equal to E, and AG to F, it ſhall be as AB is to CD, ſo is CH to AG. Therefore, the Sides that are about the equal Angles of the Parallelograms BG, DH, are reciprocal; and ſince thoſe Parallelograms are equal*, that have the Sides about the equal Angles reciprocal, therefore the Parallelogram BG is equal to the Parallelogram DH. But the Parallelogram BG is equal to that contained under AB and F; for AG is equal to F, and the Parallelogram DH equal to that contained under CD and E, ſince CH is equal to E. Therefore the Rectangle contained under AB and F, is equal to that contained under CD and E.

* 14 *of this.*

And if the Rectangle contained under AB and F, be equal to the Rectangle contained under CD and E; I ſay, the four Right Lines are Proportionals, *viz.* as AB is to CD, ſo is E to F.

For the ſame Conſtruction remaining, the Rectangle contained under AB and F, is equal to that contained under CD and E; but the Rectangle contained under AB and F, is the Rectangle BG; for AG is equal to F: And the Rectangle contained under CD and E, is the Rectangle DH; for CH is equal to E. Therefore the Parallelogram BG ſhall be equal to the Parallelogram DH, and they are equiangular; but the Sides of equal and equiangular Parallelograms, which are about the equal Angles, are *reciprocal. Wherefore, as AB is to CD, ſo is CH to AG; but CH is equal to E, and AG to F; therefore, as AB is to CD, ſo is E to F. Wherefore, *if four Right Lines be proportional, the Rectangle contained under the Extremes, is equal to the Rectangle contained under the Means; and if the Rectangle contained under the Extremes be equal to the Rectangle contained under the Means, then are the four Right Lines proportional*; which was to be demonſtrated. * 14 *of this.*

PROPOSITION XVII.

THEOREM.

If three Right Lines be proportional, the Rectangle contained under the Extremes, is equal to the Square of the Mean; and if the Rectangle under the Extremes be equal to the Square of the Mean, then the three Right Lines are proportional.

LET there be three Right Lines A, B, C, proportional; and let A be to B, as B is to C. I ſay, the Rectangle contained under A and C, is equal to the Square of B.

For, make D equal to B.

Then, becauſe A is to B, as B is to C, and B is equal to D, it ſhall be* as A is to B, ſo is D to C. But, if four Right Lines be Proportionals, the Rectangle contained under the Extremes is † equal to the Rectangle under the Means. Therefore the Rectan- * 7. 5. † 16 *of this.*

gle contained under A and C, is equal to the Rectangle under B and D; but the Rectangle under B and D is equal to the Square of D; for B is equal to D. Wherefore the Rectangle contained under A, C, is equal to the Square of B.

And, if the Rectangle contained under A, C, be equal to the Square of B; I ſay, as A is to B, ſo is B to C.

For the ſame Conſtruction remaining, the Rectangle contained under A and C, is equal to the Square of B; but the Square of B is the Rectangle contained under B, D; for B is equal to D; and the Rectangle contained under A, C, ſhall be equal to the Rectangle contained under B, D. But if the Rectangle contained under the Extremes, be equal to the Rectangle contained under the Means, the four Right Lines
‡ 16 *of this.* ſhall be † Proportionals. Therefore A is to B, as D is to C; but B is equal to D. Wherefore A is to B, as B is to C. Therefore, *if three Right Lines be proportional, the Rectangle contained under the Extremes, is equal to the Square of the Mean; and if the Rectangle under the Extremes, be equal to the Square of the Mean, then the three Right Lines are proportional;* which was to be demonſtrated.

PROPOSITION XVIII.

PROBLEM.

Upon a given Right Line, to deſcribe a Right-lined Figure, ſimilar, and ſimilarly ſituate, to a Right-lined Figure given.

LET AB be the Right Line given, and CE the Right-lined Figure. It is required to deſcribe upon the Right Line AB a Figure ſimilar, and ſimilarly ſituate, to the Right-lined Figure CE.

* 23. 1. Join DF, and make * at the Points A and B, with the Line AB, the Angles GAB, ABG, each equal to the Angles C and CDF. Whence the other Angle
† *Cor.* 32. 1. CFD is † equal to the other Angle AGB; and ſo the Triangle FCD is equiangular to the Triangle GAB; and conſequently, as FD is to GB, ſo is
‡ 4 *of this.* ‡ FC to GA; and ſo is CD to AB. Again, make the Angles BGH, GBH, at the Points B and G, with

with the Right Line BG, each equal to the Angles EFD, EDF; then the remaining Angle at E, is † equal to the remaining Angle at H. † *Cor.* 32. 1. Therefore the Triangle FDE is equiangular to the Triangle GBH; and conſequently, as FD is to GB, ſo is ‡ FE to ‡ 4 *of this.* GH; and ſo ED to HB. But it has been proved, that FD is to GB, as FC is to GA, and as CD to AB. And therefore, as FC is to AG, ſo is * CD to * 11. 5. AB; and ſo FE to GH; and ſo ED to HB. And becauſe the Angle CFD is equal to the Angle AGB; and the Angle DFE equal to the Angle BGH; the whole Angle CFE ſhall be equal to the whole Angle AGH. By the ſame Reaſon, the Angle CDE is equal to the Angle ABH; and the Angle at C equal to the Angle A; and the Angle E equal to the Angle H. Therefore the Figure AC is equiangular to the Figure CE; and they have the Sides about the equal Angles proportional. Conſequently, the Right-lined Figure AH will be † ſimilar to the Right-lined Figure CE. † *Def.* 1. *of this.* Therefore there is deſcribed upon the given Right Line AB, the Right-lined Figure AH, ſimilar, and ſimilarly ſituate, to the given Right-lined Figure CE; *which was to be done.*

PROPOSITION XIX.

THEOREM.

Similar Triangles are in the duplicate Proportion of their homologous Sides.

LET ABC, DEF, be ſimilar Triangles, having the Angle B equal to the Angle E; and let AB be to BC, as DE is to EF, ſo that BC be the Side homologous to EF. I ſay, the Triangle ABC, to the Triangle DEF, has a duplicate Proportion to that of the Side BC to the Side EF.

For, take * BG a third Proportion to BC and EF; * 11 *of this.* that is, let BC be to EF, as EF is to BG; and join GA.

Then, becauſe AB is to BC, as DE is to EF; it ſhall be (by Alternation) as AB is to DE, ſo is BC to EF; but as BC is to EF, ſo is EF to BG. Therefore, as AB is to DE, ſo is † EF to BG; conſe- † 11. 5. quently, the Sides that are about the equal Angles of the Triangles A B G, D E F, are reciprocal: But thoſe Triangles that have one Angle of the one equal to one Angle of the other, and the Sides about the equal Angles reciprocal, are ‡ equal. Therefore the Triangle ABG is equal to the Triangle DEF; and becauſe BC is to E F, as E F is to BG, and if three Right Lines be proportional, the firſt has * a duplicate Proportion to the third, of what it has to the ſecond, B C to B G ſhall have a duplicate Proportion of that which B C has to E F; and as B C is to B G, ſo is the Triangle ABC to the Triangle ABG; whence the Triangle ABC bears to the Triangle A B G a duplicate Proportion to what B C doth to E F; but the Triangle A B G is equal to the Triangle D E F. Therefore the Triangle A B C, to the Triangle DEF, ſhall be in the duplicate Proportion of that which the Side B C has to the Side E F. Wherefore *ſimilar Triangles are in the duplicate Proportion of their homologous Sides*; which was to be demonſtrated.

‡ 15 *of this.*

* *Def.* 10. 5.

Coroll. From hence it is manifeſt, if three Right Lines be proportional, then, as the firſt is to the third, ſo is a Triangle made upon the firſt to a ſimilar, and ſimilarly deſcribed Triangle upon the ſecond: Becauſe it has been proved, as C B is to B G, ſo is the Triangle A B C to the Triangle A B G, that is, to the Triangle D E F; *which was to be demonſtrated.*

PROPOSITION XX.

THEOREM.

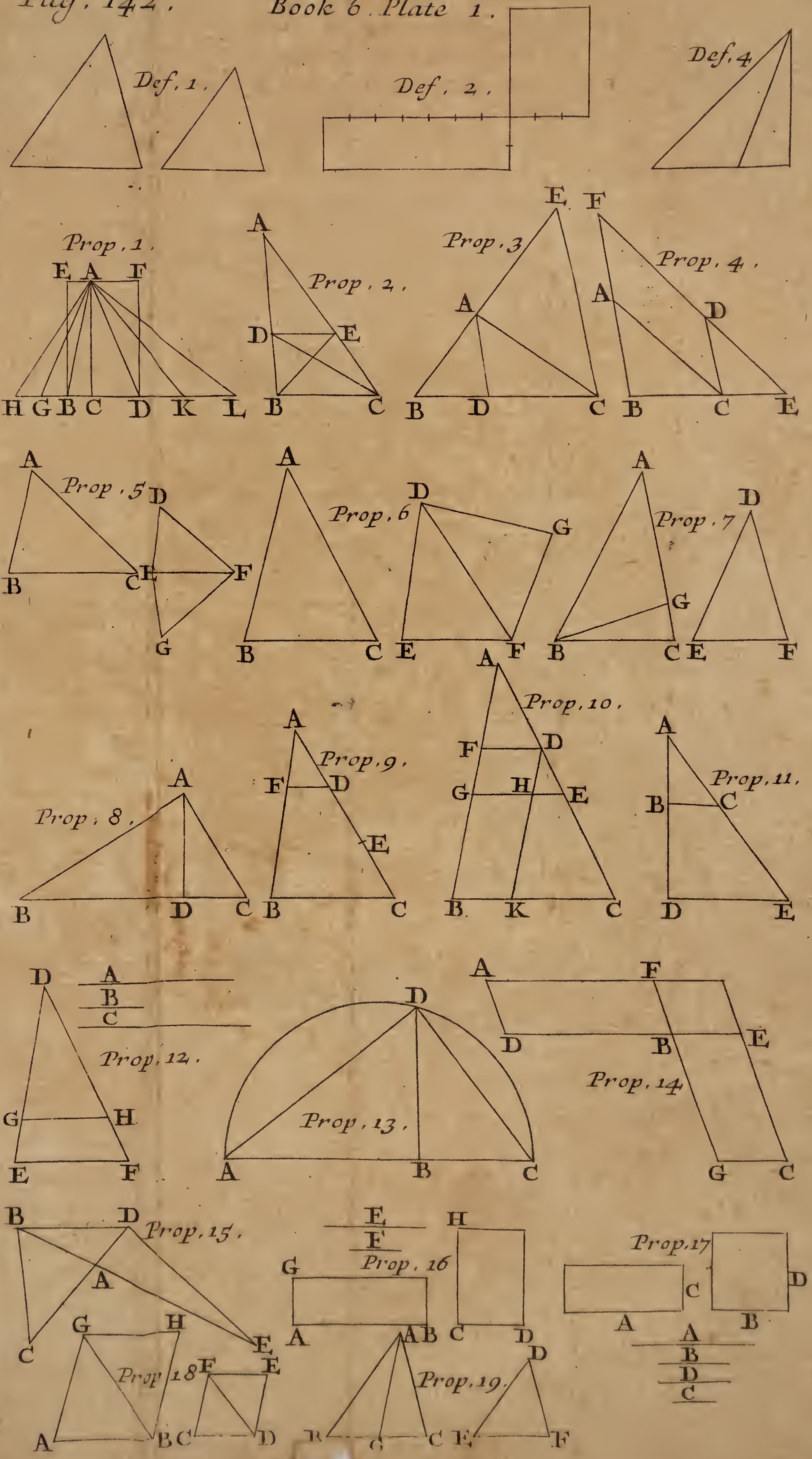
Similar Polygons are divided into ſimilar Triangles, equal in Number, and homologous to the Wholes; and Polygon to Polygon, is in the duplicate Proportion of that which one homologous Side has to the other.

LET ABCDE, FGHKL, be ſimilar Polygons; and let the Side A B be homologous to the Side F G. I ſay, the Polygons ABCDE, FGHKL, are divided into equal Numbers of ſimilar Triangles, and homologous to the Wholes; and the Polygon ABCDE, to the Polygon FGHKL, is in the duplicate Proportion of that which the Side AB has to the Side F G.

For, let B E, E C, G L, L H, be joined.

Then,

Pag. 142. *Book 6. Plate 1.*



Then, becauſe the Polygon ABCDE is ſimilar to the Polygon FGHKL, the Angle BAE is equal to the Angle GFL; and BA is to AE as GF is to FL. Now, ſince ABE, FGL, are two Triangles, having one Angle of the one equal to one Angle of the other, and the Sides about the equal Angles proportional; the Triangle ABE will be * equiangular to the Triangle FGL; and alſo ſimilar to it. Therefore the Angle ABE is equal to the Angle FGL; but the whole Angle ABC is † equal to the whole Angle FGH, becauſe of the Similarity of the Polygons. Therefore the remaining Angle EBC is equal to the remaining Angle LGH: And ſince, (by the Similarity of the Triangles ABE, FGL) as EB is to BA, ſo is LG to GF: And ſince alſo, by the Similarity of the Polygons, AB is to BC, as FG is to GH; it ſhall be ‡ by Equality of Proportion, as EB is to BC, ſo is LG to GH, that is, the Sides about the equal Angles EBC, LGH, are proportional. Wherefore the Triangle EBC is equiangular to the Triangle LGH; and conſequently alſo ſimilar to it. For the ſame Reaſon, the Triangle ECD is likewiſe ſimilar to the Triangle LHK; therefore the ſimilar Polygons ABCDE, FGHKL, are divided into equal Numbers of ſimilar Triangles.

* 6 *of this.*

† *Def.* 1. *of this.*

‡ 22. 5.

I ſay, they are alſo homologous to the Wholes, that is, That the Triangles are proportional; and the Antecedents are ABE, EBC, ECD; and their Conſequents FGL, LGH, LHK. And the Polygon ABCDE, to the Polygon FGHLK, is in the duplicate Proportion of an homologous Side of the one, to an homologous Side of the other, that is, AB to FG.

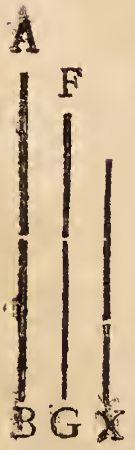
For, becauſe the Triangle ABE is ſimilar to the Triangle FGL, the Triangle ABE ſhall be * to the Triangle FGL, in the duplicate Proportion of BE to GL: For the ſame Reaſon, the Triangle BEC, to the Triangle GLH, is * in a duplicate Proportion of BE to GL: Therefore the Triangle ABE is † to the Triangle FGL, as the Triangle BEC is to the Triangle GLH. Again, becauſe the Triangle EBC is ſimilar to the Triangle LGH; the Triangle EBC, to the Triangle LGH, ſhall be in the duplicate Proportion of the Right Line CE to the Right Line HL; and ſo likewiſe the Triangle ECD, to the Tri-

* 19 *of this.*

† 11. 5.

angle L H K, ſhall be in the duplicate Proportion of C E to H L. Therefore the Triangle B E C is to the Triangle L G H, as the Triangle C E D is to the Triangle L H K. But it has been proved, that the Triangle E B C is to the Triangle L G H, as the Triangle A B E is to the Triangle F G L: Therefore, as the Triangle A B E is to the Triangle F G L, ſo is the Triangle B E C to the Triangle G H L; and ſo is the Triangle E C D to the Triangle L H K. But as one of the Antecedents is to one of the Conſe- ‡ 12. 5. quents, ſo are ‡ all the Antecedents to all the Conſequents. Wherefore, as the Triangle A B E is to the Triangle F G L, ſo is the Polygon A B C D E to the Polygon F G H K L: But the Triangle A B E, to the Triangle F G L, is in the duplicate Proportion of the homologous Side A B to the homologous Side F G; for ſimilar Triangles are in the duplicate Proportion of the homologous Sides. Wherefore the Polygon A B C D E, to the Polygon F G H K L, is in the duplicate Proportion of the homologous Side A B to the homologous Side F G. Therefore *ſimilar Polygons are divided into ſimilar Triangles, equal in Number, and homologous to the Wholes; and Polygon to Polygon, is in the duplicate Proportion of that which one homologous Side has to the other*; which was to be demonſtrated.

It may be demonſtrated after the ſame manner, that ſimilar quadrilateral Figures are to each other in the duplicate Proportion of their homologous Sides; and this has been already proved in Triangles.



Coroll. 1. Therefore univerſally ſimilar Right-lin'd Figures are to one another in the duplicate Proportion of their homologous Sides; and if X be taken a third Proportional to A B and F G, then A B will have to X a duplicate Proportion of that which A B has to F G; and a Polygon to a Polygon, and a quadrilateral Figure to a quadrilateral Figure, will be in the duplicate Proportion of that which one homologous Side has to the other; that is, A B to F G; but this has been proved in Triangles.

2. Therefore univerſally it is manifeſt, if three Right Lines be proportional, as the firſt is to the third, ſo is a Figure deſcribed upon the firſt, to a ſimilar and

ſimilarly deſcribed Figure on the ſecond; *which was to be demonſtrated.*

PROPOSITION XXI.

THEOREM.

Figures that are ſimilar to the ſame Right-lined Figure, are alſo ſimilar to one another.

LET each of the Right-lined Figures A, B, be ſimilar to the Right-lined Figure C. I ſay, the Right-lined Figure A is alſo ſimilar to the Right-lin'd Figure B.

For, becauſe the Right-lined Figure A is ſimilar to the Right-lined Figure C, it ſhall be * equiangular thereto; and the Sides about the equal Angles proportional. Again, becauſe the Right-lined Figure B is ſimilar to the Right-lined Figure C, it ſhall * be equiangular thereto; and the Sides about the equal Angles will be proportional. Therefore each of the Right-lined Figures A, B, are equiangular to C, and they have the Sides about the equal Angles proportional. Wherefore the Right-lin'd Figure A is equiangular to the Right-lin'd Figure B; and the Sides about the equal Angles are proportional; wherefore A is ſimilar to B; *which was to be demonſtrated.*

* *1 Def. of this.*

PROPOSITION XXII.

THEOREM.

If four Right Lines be proportional, the Right-lined Figures, ſimilar and ſimilarly deſcribed upon them, ſhall be proportional; and if the ſimilar Right-lin'd Figures ſimilarly deſcribed upon the Lines, be proportional, then the Right Lines ſhall be alſo proportional.

LET four Right Lines AB, CD, EF, GH, be proportional; and as AB is to CD, ſo let EF be to GH.

Now, let the ſimilar Figures KAB, LCD, be ſimilarly deſcribed * upon AB, CD; and the ſimilar Figures

* *18 of this.*

Figures MF, NH, ſimilarly deſcribed upon the Right Lines EF, GH. I ſay, as the Right-lined Figure KAB is to the Right-lin'd Figure LCD, ſo is the Right-lin'd Figure MF to the Right-lined Figure NH.

* 11 *of this.* For, take * X a third Proportional to AB, CD; and
O a third Proportional to EF, GH.

Then, becauſe AB is to CD, as EF is to GH, and
† 22. 5. as CD is to X, ſo is GH to O; it ſhall be † by
Equality of Proportion, as AB is to X, ſo is EF to O.
‡ *Cor.* 20. *of this.* But AB is to X, as the Right-lin'd Figure KAB is ‡ to
the Right-lin'd Figure LCD; and as EF is to O, ſo
is ‡ the Right-lin'd Figure MF, to the Right-lin'd
Figure NH. Therefore, as the Right-lin'd Figure
* 11. 5. KAB is to the Right-lin'd Figure LCD, ſo is * the Right-
lin'd Figure MF to the Right-lin'd Figure NH.

And, if the Right-lin'd Figure KAB be to the Right-lin'd Figure LCD, as the Right-lin'd Figure MF is to the Right-lin'd Figure NH; I ſay, as AB is to CD, ſo is EF to GH.

† 12 *of this.* For, make † EF to PR, as AB is to CD, and de-
ſcribe upon PR a Right-lin'd Figure SR ſimilar, and
alike ſituate, to either of the Figures MF and NH.

Then, becauſe AB is to CD, as EF is to PR, and
there are deſcribed upon AB, CD, ſimilar and alike
ſituate Right-lined Figures KAB, LCD, and upon
EF, PR, ſimilar and alike ſituate Figures MF, SR;
it ſhall be (by what has been already proved) as the
Right-lin'd Figure KAB is to the Right-lin'd Figure
LCD, ſo is the Right-lined Figure MF to the Right-
lin'd Figure RS: But (by the *Hyp.*) as the Right-lin'd
Figure KAB is to the Right-lin'd Figure LCD, ſo
is the Right-lin'd Figure MF to the Right-lin'd Fi-
gure NH. Therefore, as the Right-lin'd Figure MF
is to the Right-lin'd Figure NH, ſo is the Right-lin'd
Figure MF to the Right-lin'd Figure SR: And ſince
the Right-lin'd Figure MF has the ſame Proportion
to NH, as it hath to SR, the Right-lin'd Figure NH
ſhall be ‡ equal to the Right-lin'd Figure SR; it is
‡ 9. 5. alſo ſimilar to it, and alike deſcribed; therefore GH
is equal to PR. And, becauſe AB is to CD, as EF
is to PR; and PR is equal to GH, it ſhall be as AB is
to CD, ſo is EF to GH. Therefore, *if four Right
Lines be proportional, the Right-lin'd Figures, ſimilar
and ſimilarly deſcribed upon them, ſhall be proportional;
and*

and if the ſimilar Right-lined Figures, ſimilarly deſcribed upon the Lines, be proportional, then the Right Lines ſhall alſo be proportional; which was to be demonſtrated.

LEMMA.

Any three Right Lines A, B, *and* C, *being given, the Ratio of the firſt* A, *to the third* C, *is equal to the Ratio compounded of the Ratio of the firſt* A *to the ſecond* B, *and of the Ratio of the ſecond* B *to the third* C.

FOR *Example, let the Number* 3 *be the Exponent, or Denominator of the Ratio of* A *to* B; *that is, let* A *be three times* B, *and let the Number* 4 *be the Exponent of the Ratio of* B *to* C; *then the Number* 12 *produced by the Multiplication of* 4 *and* 3, *is the compounded Exponent of the Ratio of* A *to* C: *For, ſince* A *contains* B *thrice, and* B *contains* C *four times,* A *will contain* C *thrice four times, that is,* 12 *times. This is alſo true of other Multiples, or Submultiples; but this Theorem may be univerſally demonſtrated thus: The Quantity of the Ratio of* A *to* B, *is the Number* $\frac{A}{B}$ viz. *which multiplying the Conſequent, produces the Antecedent. So likewiſe the Quantity of the Ratio of* B *to* C, *is* $\frac{B}{C}$. *And theſe two Quantities multiplied by each other, produce the Number* $\frac{A\times B}{B\times C}$, *which is the Quantity of the Ratio, that the Rectangle comprehended under the Right Lines* A *and* B, *has to the Rectangle comprehended under the Right Lines* B *and* C; *and ſo the ſaid Ratio of the Rectangle under* A *and* B, *to the Rectangle under* B *and* C, *is that which, in the Senſe of* Def. 5. *of this Book, is compounded of the Ratio's of* A *to* B, *and* B *to* C; *but (by* 1. 6.) *the Rectangle contained under* A *and* B, *is to the Rectangle contained under* B *and* C, *as* A *is to* C; *therefore the Ratio of* A *to* C *is equal to the Ratio compounded of the Ratio's of* A *to* B, *and of* B *to* C.

If any four Right Lines A, B, C, *and* D, *be propoſed, the Ratio of the firſt* A *to the fourth* D, *is equal to the Ratio compounded of the Ratio of the firſt* A *to the ſecond* B, *and of the Ratio of the ſecond* B *to the third* C, *and of the Ratio of the third* C *to the fourth* D.

For, in three Right Lines A, C, *and* D, *the Ratio of* A *to* D, *is equal to the Ratio's compounded of the Ratio's of* A *to* C, *and of* C *to* D; *and it has been already demonſtrated, that the Ratio of* A *to* C *is equal to the Ratio compounded of the Ratio's of* A *to* B, *and* B *to* C. *Therefore the Ratio of* A *to* D *is equal to the Ratio compounded of the Ratio's of* A *to* B, *of* B *to* C, *and of* C *to* D. *After the ſame manner we demonſtrate, in any Number of Right Lines, that the Ratio of the firſt to the laſt is equal to the Ratio compounded of the Ratio's of the firſt to the ſecond, of the ſecond to the third, of the third to the fourth, and ſo on to the laſt.*

This is true of any other Quantities beſides Right Lines, which will be manifeſt, if the ſame Number of Right Lines A, B, C, &c. *as there are Magnitudes, be aſſumed in the ſame Ratio,* viz. *ſo that the Right Line* A *is to the Right Line* B, *as the firſt Magnitude is to the ſecond, and the Right Line* B *to the Right Line* C, *as the ſecond Magnitude is to the third, and ſo on. It is manifeſt (by* 22. 5.) *by Equality of Proportion, that the firſt Right Line* A *is to the laſt Right Line, as the firſt Magnitude is to the laſt; but the Ratio of the Right Line* A *to the laſt Right Line, is equal to the Ratio compounded of the Ratio's of* A *to* B, B *to* C, *and ſo on to the laſt Right Line: But (by the* Hyp.) *the Ratio of any one of the Right Lines to that neareſt to it, is the ſame as the Ratio of a Magnitude of the ſame Order to that neareſt it. And therefore the Ratio of the firſt Magnitude to the laſt, is equal to the Ratio compounded of the Ratio's of the firſt Magnitude to the ſecond, of the ſecond to the third, and ſo on to the laſt;* which was to be demonſtrated.

PROPOSITION XXIII.

THEOREM.

Equiangular Parallelograms have the Proportion to one another that is compounded of their Sides.

LET AC, CF, be equiangular Parallelograms, having the Angle BCD equal to the Angle ECG. I ſay, the Parallelogram AC, to the Parallelogram CF, is in the Proportion compounded of their Sides, *viz.* compounded of the Proportion of BC to CG, and of DC to CE.

For, let BC be placed in the ſame Right Line with CG.

Then DC ſhall be * in a ſtrait Line with CE, * 14. 1.
and complete the Parallelogram DG; and then†, as † 12 *of this.*
BC is to CG, ſo is ſome Right Line K to L; and as DC is to CE, ſo let L be to M.

Then the Proportions of K to L, and of L to M, are the ſame as the Proportions of the Sides, *viz.* of BC to CG, and DC to CE; but the Proportion of
K to M is ‡ compounded of the Proportion of K ‡ *Lemma*
to L, and of the Proportion of L to M. Wherefore *preced.*
alſo K to M hath a Proportion compounded of the Sides. Then, becauſe BC is to CG as the Parallelo-
gram AC is * to the Parallelogram CH: And ſince * 1 *of this.*
BC is to CG, as K is to L, it ſhall be † as K is to L, † 11. 5.
ſo is the Parallelogram AC, to the Parallelogram CH. Again, becauſe DC is to CE, as the Parallelogram CH is, to the Parallelogram CF; and ſince as DC is to CE, ſo is L to M. Therefore, as L is to M, ſo ſhall † the Parallelogram CH be to the Parallelogram CF; and conſequently, ſince it has been prov'd that K is to L, as the Parallelogram AC is to the Parallelogram CH, and as L is to M, ſo is the Parallelogram CH to the Parallelogram CF; it ſhall
be ‡ by Equality of Proportion, as K is to M, ſo is ‡ 22. 5.
the Parallelogram AC to the Parallelogram CF; but K to M hath a Proportion compounded of the Sides: Therefore alſo the Parallelogram AC, to the Parallelogram CF, hath a Proportion compounded of the Sides. Wherefore *equiangular Parallelograms have*

the Proportion to one another that is compounded of their Sides; which was to be demonſtrated.

PROPOSITION XXIV.

THEOREM.

In every Parallelogram, the Parallelograms that are about the Diameter, are ſimilar to the Whole, and alſo to one another.

LET ABCD be a Parallelogram, whoſe Diameter is AC; and EG, HK, be Parallelograms about the Diameter AC. I ſay, the Parallelograms EG, HK, are ſimilar to the Whole ABCD, and alſo to each other.

For, becauſe EF is drawn parallel to BC, the Side of the Triangle ABC, it ſhall be * as BE to EA, ſo is CF to FA. Again, becauſe FG is drawn parallel to CD, the Side of the Triangle ACD, it ſhall be as CF to FA, ſo is * DG to GA. But CF is to FA, (as has been proved) as BE is to EA. Therefore, as BE is to EA, ſo is † DG to GA; and by compounding, as BA is to AE, ſo is ‡ DA to AG; and by Alternation, as BA is to AD, ſo is EA to AG. Therefore the Sides of the Parallelograms ABCD, EG, which are about the common Angle BAD, are proportional. And becauſe GF is parallel to DC, the Angle AGF is * equal to the Angle ADC, and the Angle GFA equal to the Angle DCA; and the Angle DAC is common to the two Triangles ADC, AGF. Wherefore the Triangle ADC will be equiangular to the Triangle AGF. For the ſame Reaſon, the Triangle ACB is equiangular to the Triangle AFE. Therefore the whole Parallelogram ABCD is equiangular to the Parallelogram EG; and ſo AD is to DC, as AG is † to GF. But DC is to CA, as GF is to FA; and AC is to CB, as AF is to FE; and moreover, CB is to BA, as FE is to EA. Wherefore, ſince it has been proved, that DC is to CA, as GF is to FA; and AC is to CB, as AF is to FE; it ſhall be, by Equality of Proportion, as DC is to CB, ſo is GF to FE. Therefore the Sides that are about the equal Angles of the Parallelograms ABCD,

* 2 *of this.*
† 11. 5.
‡ 18. 5.
* 29. 9.
† 4 *of this.*

ABCD, EG, are proportional; and accordingly the Parallelogram ABCD is ſimilar to the Parallelogram EG. For the ſame Reaſon the Parallelogram ABCD is ſimilar to the Parallelogram KH. Therefore both the Parallelograms EG, HK, are ſimilar to the Parallelogram ABCD. But Right-lin'd Figures that are ſimilar to the ſame Right-lin'd Figure, are ſimilar to one another. *21 *of this.* Therefore the Parallelogram EG is ſimilar to the Parallelogram HK. And ſo *in every Parallelogram, the Parallelograms that are about the Diameter are ſimilar to the Whole, and alſo to one another*; which was to be demonſtrated.

PROPOSITION XXV.

PROBLEM.

To deſcribe a Right-lin'd Figure ſimilar to a Right-lin'd Figure which ſhall be given, and equal to another Right-lin'd Figure given.

LET ABC be a given Right-lined Figure, to which it is required to deſcribe another ſimilar and equal to D.

On the Side BC of the given Figure ABC, * make the Parallelogram BE equal to the Right-lin'd Figure ABC; and on the Side CE make * the Parallelogram CM equal to the Right-lin'd Figure D, in the Angle FCE, equal to the Angle CBL. Then BC, CF, as alſo LE, EM, will be † in two ſtrait Lines. Find ‡ GH a mean Proportional between BC, CF, and on GH let there be deſcribed * the Right-lined Figure KGH, ſimilar, and alike ſituate, to the Right-lined Figure ABC.

* 44. 1.
† 14. 1.
‡ 13 *of this.*
* 18 *of this.*

And then, becauſe BC is to GH, as GH is to CF, and ſince, when three Right Lines are proportional, the firſt is to the third as the Figure deſcribed on the firſt is † to a ſimilar and alike ſituate Figure deſcribed on the ſecond, it ſhall be as BC is to CF, ſo is the Right-lined Figure ABC to the Right-lined Figure KGH. But as BC is to CF, ſo is ‡ the Parallelogram BE to the Parallelogram EF. Therefore, as the Right-lined Figure ABC is to the Right-lin'd Figure KGH, ſo is the Parallelogram BE to the Parallelogram EF.

† *Cor.* 20. *of this.*
‡ 1 *of this.*

Wherefore, (by Alternation) as the Right-lined Figure ABC is to the Parallelogram BE, ſo is the Right-lined Figure KGH to the Parallelogram EF. But the Right-lined Figure ABC is equal to the Parallelogram BE. Therefore the Right-lined Figure KGH is alſo equal to the Parallelogram EF. But the Parallelogram EF is equal to the Right-lined Figure D. Therefore the Right-lin'd Figure KGH is equal to D. But KGH is ſimilar to ABC. Conſequently, there is deſcribed the Right-lin'd Figure KGH ſimilar to the given Figure ABC, and equal to the given Figure D; *which was to be done.*

PROPOSITION XXVI.

THEOREM.

If from a Parallelogram be taken away another ſimilar to the Whole, and in like manner ſituate, having alſo an Angle common with it, then is that Parallelogram about the ſame Diameter with the Whole.

LET the Parallelogram AF be taken away from the Parallelogram ABCD, ſimilar to ABCD, and in like manner ſituate, having the Angle DAB common. I ſay, the Parallelogram ABCD is about the ſame Diameter with the Parallelogram AF.

For, if it be not, let AHC be the Diameter of the Parallelogram BD, and let GF be produced to H; alſo let HK be drawn parallel to AD, or BC.

Then, becauſe the Parallelogram ABCD is about
the ſame Diameter as the Parallelogram KG, the Pa-
* 24 *of this.* rallelogram ABCD ſhall be * ſimilar to the Paralle-
† *Def.* 1. *of this.* logram KG; and ſo, as DA is to AB, ſo is † GA to
AK. But becauſe of the Similarity of the Parallelograms ABCD, EG, as DA is to AB, ſo is GA to
‡ 11. 5. AE. And therefore, as GA is ‡ to AE, ſo is GA
to AK. And ſince GA has the ſame Proportion to
∔ 9. 5. AK as to AE, AE is ∔ equal to AK, the leſs to a
greater, which is abſurd. Therefore the Parallelogram ABCD is not about the ſame Diameter as the Parallelogram AH. And therefore it will be about the ſame Diameter with the Parallelogram AF. Therefore, *if from a Parallelogram be taken away another ſimilar*

to the Whole, and in like manner ſituate, having alſo an Angle common with it, then is that Parallelogram about the ſame Diameter with the Whole; which was to be demonſtrated.

PROPOSITION XXVII.

THEOREM.

Of all Parallelograms applied to the ſame Right Line, and wanting in Figure by Parallelograms ſimilar and alike ſituate, deſcribed on the half Line, the greateſt is that which is applied to the half Line, being ſimilar to the Defect.

LET AB be a Right Line, biſected in the Point C, and let the Parallelogram AD be applied to the Right Line AB, wanting in Figure the Parallelogram CE, ſimilar and alike ſituate to that deſcribed on half of the Right Line AB. I ſay, AD is the greateſt of all Parallelograms applied to the Right Line AB, wanting in Figure by Parallelograms ſimilar and alike ſituate to CE. For, let the Parallelogram AF be applied to the Right Line AB, wanting in Figure the Parallelogram HK, ſimilar and alike ſituate to the Parallelogram CE. I ſay, the Parallelogram AD is greater than the Parallelogram AF.

For, becauſe the Parallelogram CE is ſimilar to the Parallelogram HK, they ſtand * about the ſame Diameter; let DB their Diameter be drawn, and the Figure deſcribed. Then, ſince the Parallelogram CF is † equal to FE, let HK, which is common, be added; and the Whole CH is equal to the Whole KE. But CH is ‡ equal to CG, becauſe the Right Line AC is equal to CB. Therefore the Whole AF is equal to the Gnomon LNM; and ſo CE, that is, the Parallelogram AD, is greater than the Parallelogram AF. Therefore, *of all Parallelograms applied to the ſame Right Line, and wanting in Figure by Parallelograms ſimilar and alike ſituate, deſcribed on the half Line, the greateſt is that which is applied to the half Line, being ſimilar to the Defect;* which was to be demonſtrated.

* 26 *of this.*
† 43. 1.
‡ 36. 1.

PROPOSITION XXVIII.

PROBLEM.

To a Right Line given to apply a Parallelogram equal to a Right-lined Figure given, deficient by a Parallelogram, which is ſimilar to another given Parallelogram; but it is neceſſary, that the Right-lined Figure given, to which the Parallelogram to be applied muſt be equal, be not greater than the Parallelogram which is applied to the half Line; ſince the Defects muſt be ſimilar, viz. *the Defect of the Parallelogram applied to the half Line, and the Defect of the Parallelogram to be applied.*

LET AB be a given Right Line, and let the given Right-lin'd Figure, to which the Parallelogram to be applied to the Right Line AB muſt be equal, be C, which muſt not be greater than the Parallelogram applied to the half Line, the Defects being ſimilar; and let D be the Parallelogram, to which the Defect of the Parallelogram to be applied is ſimilar. Now it is requir'd to apply a Parallelogram equal to the given Right-lined Figure C to the given Right Line AB, deficient by a Parallelogram ſimilar to D.

* 18 *of this.* Let AB be biſected in E, and on EB deſcribe * the Parallelogram EBFG, ſimilar and alike ſituate to D, and complete the Parallelogram AG.

Now AG is either equal to C, or greater than it, becauſe of the Determination. If AG be equal to C, what was propoſed will be done; for the Parallelogram AG is applied to the Right Line AB, equal to the given Right-lined Figure C, deficient by the Parallelogram EF, ſimilar to the Parallelogram D. But if it be not equal, then HE is greater than C; but EF is equal to HE. Therefore EF ſhall alſo be greater than C. Now make † the Parallelogram KLMN ſimilar and alike ſituate to D, and equal to the Exceſs, by which EF exceeds C. But D is ſimilar to EF: Wherefore KM ſhall alſo be ſimilar to EF. Therefore let the Right Line KL be homologous to GE, and LM to GF. Then, becauſe EF is equal to C and KM together, EF will be greater than KM; † 25 *of this.*

and ſo the Right Line GE is greater than KL, and GF than LM. Make GX equal to KL, and GO equal to LM, and complete the Parallelogram XGOP. Therefore XO is equal and ſimilar to KM; but KM is ſimilar to EF; therefore XO is * ſimilar to EF, and ſo XO is † about the ſame Diameter with FE: Let GPB be their Diameter, and the Figure be deſcribed. * 21 *of this.* † 26 *of this.*

Then, ſince EF is equal to C and KM together, and XO is equal to KM, the Gnomon ΥΦΨ remaining is equal to the remaining Figure C; and becauſe OR is equal to XS, let SR, which is common, be added; then the Whole OB is equal to the Whole XB; but XB is equal to TE, ſince the Side AE is equal to the Side EB. Wherefore TE is equal to OB. Add XS, which is is common, and then the whole TS is equal to the whole Gnomon ΥΦΨ; but the Gnomon ΥΦΨ has been proved equal to C; and TS ſhall be equal to C; and ſo the Parallelogram TS is applied to the Right Line AB, equal to the given Right-lin'd Figure C, and deficient by a Parallelogram SR, ſimilar to the Parallelogram D, becauſe SR is ſimilar to FE; *which was to be done.*

PROPOSITION XXIX.

THEOREM.

To a Right Line given, to apply a Parallelogram equal to a Right-lin'd Figure given, exceeding by a Parallelogram, which ſhall be ſimilar to another given Parallelogram.

LET AB be a given Right Line, and let C be the given Right-lin'd Figure, to which that to be applied to AB muſt be equal. Likewiſe let D be the Parallelogram, to which the exceeding Parallelogram is to be ſimilar; it is requir'd to apply a Parallelogram to the Right Line AB, equal to the given Right-lined Figure C, exceeding by a Parallelogram ſimilar to D.

Biſect AB in E, and let the Parallelogram EL be deſcribed * upon the Right Line EB, ſimilar and alike ſituate to D; and that † the Parallelogram GH equal * 18. I. † 25 *of this.*

to E L and C together, but ſimilar to D, and alike ſituate. Therefore G H is ſimilar to E L; let K H be a Side homologous to F L, and K G to F E. Then, becauſe the Parallelogram G H is greater than the Parallelogram E L, the Right Line K H will be greater than F L, and K G greater than F E. Let F L, F E, be produced, and let F L M be equal to K H, F E N equal to K G, and complete the Parallelogram M N. Therefore M N is equal and ſimilar to G H; but G H ‡ 1 *of this.* is ſimilar to E L, and ſo M N ſhall be ‡ ſimilar to * 26 *of this.* E L; and accordingly E L is * about the ſame Diameter with M N. Let F X be their Diameter, and deſcribe the Figure.

Then, ſince G H is equal to E L and C together, as likewiſe to M N; therefore M N ſhall be equal to E L and C. Let E L, which is common, be taken away; then the Gnomon ϒΦΨ remaining is equal to C; and ſince A E is equal to E B, the Parallelogram A N will be alſo equal to the Parallelogram E P, that is, to L O; and if E X, which is common, be added, then the whole Parallelogram A X is equal to the Gnomon ϒΦΨ; but the Gnomon ϒΦΨ is equal to C. Therefore A X ſhall be alſo equal to C. Wherefore the Parallelogram A X is applied to the given Right Line A B, equal to the given Right-lin'd Figure C, and exceeding by the Parallelogram P O, ſimilar to the Parallelogram D; *which was to be done.*

PROPOSITION XXX.

PROBLEM.

To cut a given terminate Right Line according to extreme and mean Ratio.

LET A B be a given terminate Line; it is requir'd to cut the ſame according to extreme and mean Ratio.

* 46. 1. Deſcribe * B C the Square of A B, and apply the Parallelogram C D to A C, equal to the Square B C, † 29 *of this.* exceeding † by the Figure A D ſimilar to B C; but B C is a Square; therefore A D ſhall alſo be a Square.

Now, becauſe B C is equal to C D, take away C E, which is common; then B F remaining ſhall be equal to AD remaining; but BF is equiangular to AD; therefore the Sides that are about the equal Angles are ‡ reciprocally proportional; and ſo, as FE is to ED, ſo is AE to EB; but FE is * equal to AC, that is, to AB, and ED to AE. Wherefore as BA is to AE, ſo is AE to EB; but AB is greater than AE; therefore AE is † greater than EB; and ſo the Right Line AB is cut according to the extreme and mean Ratio in the Point E; and AE is the greater Segment thereof; *which was to be done.* ‡ 14 *of this.* * 34. 1. † 14. 5.

Otherwiſe thus: Let AB be the Right Line given; it is required to cut the ſame into extreme and mean Ratio.

Divide ‡ AB ſo in C, that the Rectangle contained under AB, BC, be equal to the Square of AC. ‡ 11. 2.

Then, becauſe the Rectangle under AB, BC, is equal to the Square of AC, it ſhall be * as BA is to AC, ſo is AC to CB; and ſo the Right Line AB is cut into mean and extreme Ratio; *which was to be done.* * 17 *of this.*

PROPOSITION XXXI.

THEOREM.

Any Figure deſcribed upon the Side of a Right-angled Triangle ſubtending the Right Angle, is equal to the Figures deſcribed upon the Sides containing the Right Angle, being ſimilar and alike ſituate to the former Figure.

LET ABC be a rectangular Triangle, having the Right Angle BAC. I ſay, the Figure deſcribed on BC, is equal to the two Figures together deſcribed on BA, AC, which are ſimilar and alike ſituate to the Figure deſcribed on BC.

For, draw the Perpendicular AD.

Then, becauſe the Right-line AD is drawn in the Right-angled Triangle ACB, from the Right Angle A, perpendicular to the Baſe BC; the Triangles ABD, ADC, which are about the Perpendicular AD, will be * ſimilar to the whole Triangle ABC, and alſo to each other. Then, becauſe the Triangle ABC is ſimilar to the Triangle ABD, it ſhall be * as CB is to BA, * 8 *of this.*

ſo is BA to BD: And ſince when three Right Lines are proportional, the firſt ſhall be † to the third, as a Figure deſcribed on the firſt, to a ſimilar and alike ſituate Figure deſcribed on the ſecond; therefore, as CB is to BD, ſo is a Figure deſcribed on CB to a ſimilar and alike ſituate Figure deſcribed on BA. For the ſame Reaſon as BC is to CD, ſo is a Figure deſcribed on BC to one deſcribed on CA. Wherefore alſo, as BC is to BD and DC together, ſo is ‡ the Figure deſcribed on BC, to thoſe two together, that are deſcribed ſimilar and alike ſituate on BA, AC; but BC is equal to BD and DC together: Therefore the Figure deſcribed on BC is equal to thoſe together deſcribed on BA, AC, ſimilar and alike ſituate to that on BC. Wherefore *any Figure deſcribed upon the Side of a Right-angled Triangle ſubtending the Right Angle, is equal to the Figures deſcribed upon the Sides containing the Right Angle, being ſimilar and alike ſituate to the former Figure*; which was to be demonſtrated.

† *Cor.* 20. *of this.*

‡ 24. 5.

PROPOSITION XXXII.

THEOREM.

If two Triangles having two Sides proportional to two Sides, be ſo compounded, or ſet together at one Angle, that their homologous Sides be parallel, then the other Sides of theſe Triangles will be in one ſtrait Line.

LET there be two Triangles ABC, DCE, having two Sides BA, AC, of the one, proportional to two Sides CD, DE, of the other; *viz.* Let BA be to AC, as CD is to DE; alſo let AB be parallel to DC, and AC to DE. I ſay, BC, CE, are both in one ſtrait Line.

For, becauſe AB is parallel to DC, and the Right Line AC falls on them, the alternate Angles BAC, ACD, will be * equal to each other. And by the ſame Reaſon, the Angle CDE is equal to the Angle ACD; wherefore the Angle BAC is equal to the Angle CDE. Then, becauſe ABC, DCE, are two Triangles, having one Angle A equal to one Angle

* 29. 1.

D, and the Sides about the equal Angles proportional, *viz.* BA to AC, as CE to DE; the Triangle ABC will be * equiangular to the Triangle DCE; wherefore the Angle ABC is equal to the Angle DCE; but the Angle ACD has been proved to be equal to the Angle BAC; therefore the whole Angle ACE is equal to the two Angles ABC, BAC; and if ACB, which is common, be added, then the Angles ACE, ACB, are equal to the Angles BAC, ACB, CBA; but the Angles BAC, ACB, CBA, are equal to two Right Angles. Therefore the Angles ACE, ACB, will alſo be equal to two Right Angles; and ſo at the Point C in the Right Line AC, two Right Lines BC, CE, tending contrary ways, make the adjacent Angles ACE, ACB, equal to two Right Angles; therefore BC ſhall be † in the ſame Right Line with CE. Wherefore, *if two Triangles having two Sides proportional to two Sides, be ſo compounded, or ſet together at one Angle, that their homologous Sides be parallel, then the other Sides of theſe Triangles will be in one ſtrait Line*; which was to be demonſtrated.

* 6 *of this.*

† 14, 1.

PROPOSITION XXXIII.

THEOREM.

In equal Circles the Angles have the ſame Proportion with their Circumferences on which they ſtand, whether the Angles be at the Centres, or at the Circumferences; and ſo likewiſe are the Sectors, as being at the Centres.

LET ABC, DEF, be equal Circles; and let the Angles BGC, EHF, be at their Centres G, H; and the Angles BAC, EDF, at their Circumferences. I ſay, as the Circumference BC is to the Circumference EF, ſo is the Angle BGC to the Angle EHF; and ſo is the Angle BAC to the Angle EDF; and ſo is the Sector BGC to the Sector EHF.

For, aſſume any Number of continuous Circumferences CK, KL, each equal to BC; and alſo any Number FM, MN, each equal to EF; and join GK, GL, HM, HN.

Then, becauſe the Circumferences BC, CK, KL, are equal to each other, the Angles BGC, CGK, * 27. 3. KGL, will be * alſo equal to one another; and ſo the Circumference BL is the ſame Multiple of the Circumference BC, as the Angle BGL is of the Angle BGC. For the ſame Reaſon, the Circumference NE is the ſame Multiple of the Circumference EF, as the Angle EHN is of the Angle EHF; but if the Circumference BL be equal to the Circumference EN, then the Angle BGL ſhall be equal to the Angle EHN; and if the Circumference BL be greater than the Circumference EN, the Angle BGL will be greater than the Angle EHN; and if leſs, leſs. Therefore here are four Magnitudes, *viz.* the two Circumferences BC, EF, and the two Angles BGC, EHF; and ſince there are taken Equimultiples of the Circumference BC, and the Angle BGC; to wit, the Circumference BL, and the Angle BGL; as alſo Equimultiples of the Circumference EF, and the Angle EHF, *viz.* the Circumference EN, and the Angle EHN; and becauſe it is proved, if the Circumference BL exceeds the Circumference EN, the Angle BGL will likewiſe exceed the Angle EHN; and if equal, equal; if leſs, leſs; it ſhall be as the Circumference BC is to the Circumference EF, ſo † *Def.* 5. 5. is † the Angle BGC to the Angle EHF; but as the ‡ 15. 5. Angle BGC is to the Angle EHF, ſo is ‡ the * 20. 3. Angle BAC to the Angle EDF; for the former are * double to the latter. Therefore, as the Circumference BC is to the Circumference EF, ſo is the Angle BGC to the Angle EHF; and ſo the Angle BAC to the Angle EDF.

Wherefore, in equal Circles, Angles have the ſame Proportion as the Circumferences they ſtand on, whether they be at the Centres, or at the Circumferences.

I ſay, moreover, that as the Circumference BC is to the Circumference EF, ſo is the Sector GBC to the Sector HFE.

For, join BC, CK; and aſſume the Points X, O, in the Circumferences BC, CK; and join BX, XC, CO, OK.

Then, becauſe the two Sides BG, GC, are eqnal to the two Sides CG, GK, and they contain equal † 4. 1. Angles, the Baſe BC ſhall be † equal to the Baſe

CK; as likewiſe the Triangle GBC to the Triangle GCK. And, becauſe the Circumference BC is equal to the Circumference CK, and the Circumference remaining, which makes up the whole Circle ABC, is equal to the remaining Circumference, which makes up the ſame Circle, the Angle BXC is equal to the Angle COX; and ſo the Segment BXC is ſimilar to the Segment COK; and they are upon equal Right Lines BC, CK; but ſimilar Segments of Circles that ſtand upon equal Right Lines, are * equal to each other: Therefore the Segment BXC is equal to the Segment COK. But the Triangle BGC is alſo equal to the Triangle CGK; and ſo the whole Sector BGC will be equal to the whole Sector CGK. By the ſame Reaſon the Sector GKL will be equal to the Sector GBC, or GCK; therefore the three Sectors BGC, CGK, KGL, are equal to one another; ſo likewiſe are the Sectors HEF, HFM, HMN. Wherefore the Circumference LB is the ſame Multiple of the Circumference BC, as the Sector GBL is of the Sector GBC. For the ſame Reaſon, the Circumference NE is the ſame Multiple of the Circumference EF, as the Sector HEN is of the Sector HEF; but if the Circumference BL be equal to the Circumference EN, then the Sector BGL will be equal to the Sector EHN; and if the Circumference BL exceeds the Circumference EN, then the Sector BGL will alſo exceed the Sector EHN; and if leſs, leſs. Therefore, ſince there are four Magnitudes, to wit, the two Circumferences BC, EF, and the two Sectors GBC, EHF; and there are taken of the Circumference BL, and the Sector GBL, Equimultiples of the Circumference CB, and the Sector CGB; as alſo of the Circumference EN, and the Sector HEN, Equimultiples of the Circumference EF, and the Sector HEF; and becauſe it is proved, that if the Circumference BL exceeds the Circumference EN, the Sector BGL will alſo exceed the Sector EHN; and if equal, equal; if leſs, leſs; therefore, as the Circumference BC is to the Circumference EF, ſo is the Sector GBC to the Sector HEF; *which was to be demonſtrated.*

* 24. 3.

Coroll. I.

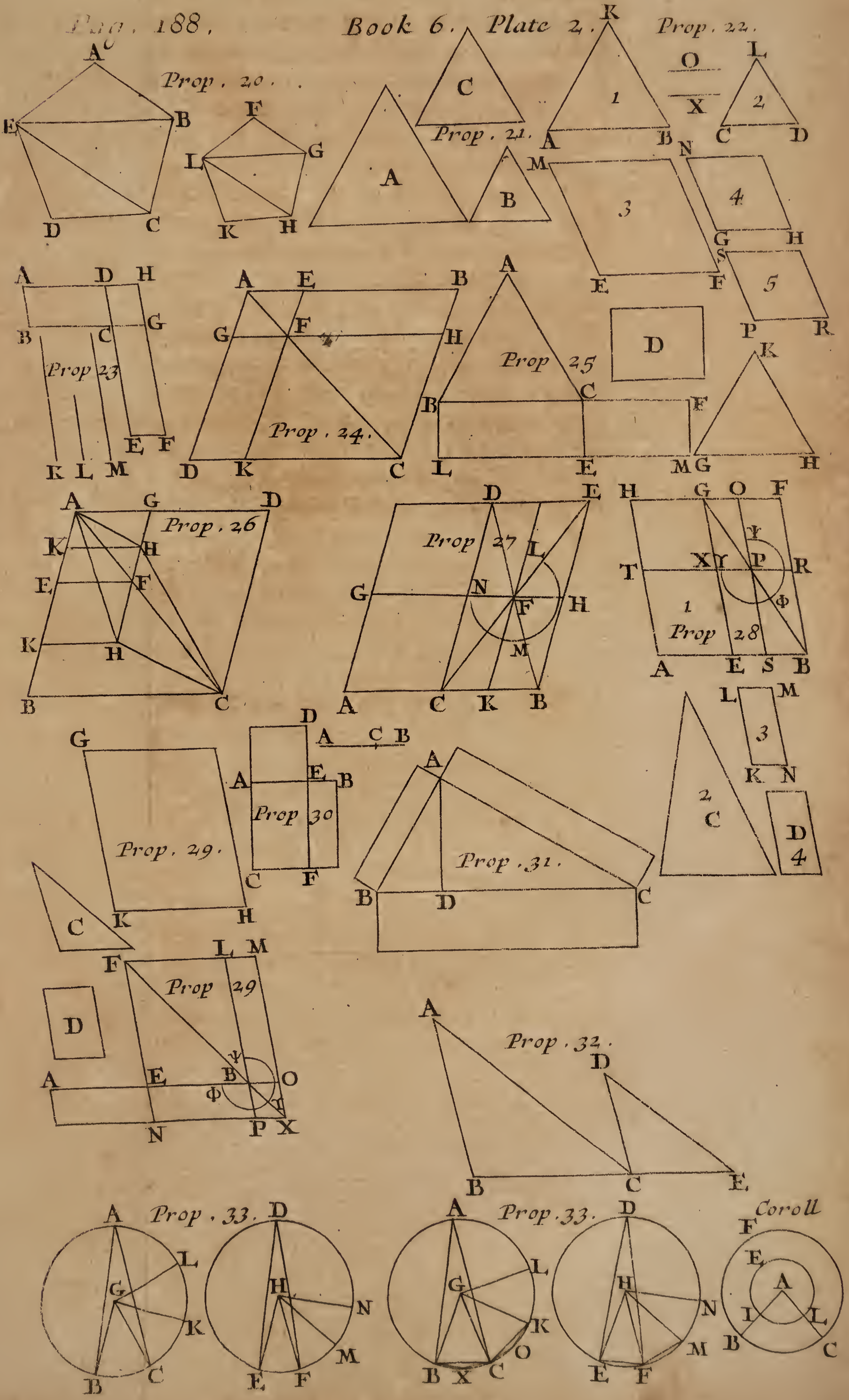
Coroll. 1. An Angle at the Centre of a Circle is to four Right Angles, as an Arc on which it ſtands is to the whole Circumference; for as the Angle BAC is to a Right Angle, ſo is the Arc BC to a Quadrant of the Circle: Wherefore, if the Conſequents be quadrupled, the Angle BAC ſhall be to four Right Angles, as the Arc BC is to the whole Circumference.

2. The Arcs IL, BC, of unequal Circles, which ſubtend equal Angles, whether at their Centres, or Circumferences, are ſimilar; for IL is to the whole Circumference ILE, as the Angle IAL is to four Right Angles; but as IAL, or BAC, is to four Right Angles, ſo is the Arc BC to the whole Circumference BCF. Therefore, as IL is to the whole Circumference ILE, ſo is BC to the whole Circumference BCF; and ſo the Arcs IL, BC, are ſimilar.

3. Two Semi-diameters AB, AC, cut off ſimilar Arcs IL, BC, from concentric Circumferences.

The END *of the* SIXTH BOOK.

Pag. 188. Book 6. Plate 2.



EUCLID's ELEMENTS.

BOOK XI.

DEFINITIONS.

I. *A Solid is that which has Length, Breadth, and Thickneſs.*

II. *The Term of a Solid is a Superficies.*

III. *A Right Line is perpendicular to a Plane, when it makes Right Angles with all the Lines that touch it, and are drawn in the ſaid Plane.*

IV. *A Plane is perpendicular to a Plane, when the Right Lines in one Plane, drawn at Right Angles to the common Section of the two Planes, are at Right Angles to the other Plane.*

V. *The Inclination of a Right Line to a Plane, is the acute Angle contained under that Line, and another Right one drawn in the Plane from that End of the inclining Line, which is in the Plane, to the Point where a Right Line falls from the other End of the inclining Line perpendicular to the Plane.*

VI. *The Inclination of a Plane to a Plane, is the acute Angle contained under the Right Lines drawn*

drawn in both the Planes to the ſame Point of their common Interſection, and making Right Angles with it.

VII. *Planes are ſaid to be inclined ſimilarly, when the ſaid Angles of Inclination are equal.*

VIII. *Parallel Planes are ſuch, which being produced never meet.*

IX. *Similar ſolid Figures are ſuch that are contained under equal Numbers of ſimilar Planes.*

X. *Equal and ſimilar ſolid Figures are thoſe that are contained under equal Numbers of ſimilar and equal Planes.*

XI. *A ſolid Angle is the Inclination of more than two Right Lines that touch one another, and are not in the ſame Superficies: Or, a ſolid Angle is that which is contained under more than two plane Angles, which are not in the ſame Superficies, but being all at one Point.*

XII. *A Pyramid is a ſolid Figure comprehended under divers Planes ſet upon one Plane, and put together at one Point.*

XIII. *A Priſm is a ſolid Figure contained under Planes, whereof the two oppoſite are equal, ſimilar, and parallel, and the others Parallelograms.*

XIV. *A Sphere is a ſolid Figure, made when the Diameter of a Semicircle remaining at Reſt, the Semicircle is turned about till it returns to the ſame Place from whence it began to move.*

XV. *The Axis of a Sphere is that fixed Line, about which the Semicircle is turned.*

XVI. *The Centre of a Sphere is the ſame with that of the Semicircle.*

XVII. *The Diameter of a Sphere is a Right Line drawn thro' the Centre, and terminated on either Side by the Superficies of the Sphere.*

XVIII. *A Cone is a Figure deſcribed, when one of the Sides of a Right-angled Triangle, containing the*

the Right Angle, remaining fixed, the Triangle is turned about till it returns to the Place from whence it firſt began to move. And if the fixed Right Line be equal to the other that contains the Right Angle, then the Cone is a rectangular Cone; but if it be leſs, it is an obtuſe-angled Cone; if greater, an acute-angled Cone.

XIX. *The Axis of a Cone is that fixed Right Line, about which the Triangle is moved.*

XX. *The Baſe of a Cone is the Circle deſcribed by the Right Line moved about.*

XXI. *A Cylinder is a Figure deſcribed by the Motion of a Right-angled Parallelogram, one of the Sides containing the Right Angle, remaining fixed while the Parallelogram is turned about to the ſame Place from whence it began to be moved.*

XXII. *The Axis of a Cylinder is that fixed Right Line, about which the Parallelogram is turned.*

XXIII. *And the Baſes of a Cylinder are the Circles that be deſcribed by the Motion of the two oppoſite Sides of the Parallelogram.*

XXIV. *Similar Cones and Cylinders are ſuch, whoſe Axes and Diameters of their Baſes are proportional.*

XXV. *A Cube is a ſolid Figure contained under ſix equal Squares.*

XXVI. *A Tetrahedron is a ſolid Figure contained under four equal equilateral Triangles.*

XXVII. *An Octahedron is a ſolid Figure contained under eight equal equilateral Triangles.*

XXVIII. *A Dodecahedron is a ſolid Figure contained under twelve equal equilateral and equiangular Pentagons.*

XXIX. *An Icoſahedron is a ſolid Figure contained under twenty equal equilateral Triangles.*

XXX. *A Parallelopipedon is a Figure contained under ſix quadrilateral Figures, whereof thoſe which are oppoſite are parallel.*

PRO-

PROPOSITION I.

THEOREM.

One Part of a Right Line cannot be in a plane Superficies, and another Part above it.

FOR, if poſſible, let the Part AB of the Right Line ABC, be in a plane Superficies, and the Part BC above the ſame.

There will be ſome Right Line in the aforeſaid Plane, which with AB will be but one ſtrait Line. Let this Line be DB.

Then the two given Right Lines ABC, ABD, have one common Segment AB, which is impoſſible; for one Right Line will not meet another in more Points than one. Wherefore *one Part of a Right Line cannot be in a plane Superficies, and another Part above it*; which was to be demonſtrated.

PROPOSITION II.

THEOREM.

If two Right Lines cut each other, they are both in one Plane, and every Triangle is in one Plane.

LET two Right Lines AB, CD, cut each other in the Point E. I ſay, they are both in one Plane, and every Triangle is in one Plane.

For, take any Points, F and G, in the Right Lines AB, CD; and join CB, FG; and let there be drawn FH, GK. In the firſt Place, I ſay, the Triangle EBC is in one Plane.

For, if one Part FHC, or GBK, of the Triangle EBC, be in one Plane, and the other Part in another Plane; then one Part of each of the Lines EC, EB, ſhall be in one Plane, and the other Part in another Plane; which we have proved * to be abſurd. Therefore the Triangle EBC is one Plane; but both the Right Lines EC, EB, are in the ſame Plane as the Triangle

* 1 *of this.*

Triangle BCE is; and AB, CD, are both in the ſame Plane as EC, EB, are. Wherefore the Right Lines AB, CD, are both in one Plane, and every Triangle is in one Plane; *which was to be demonſtrated.*

PROPOSITION III.

THEOREM.

If two Planes cut each other, their common Section will be a Right Line.

LET two Planes AB, CB, cut each other, whoſe common Section is the Line DB. I ſay, DB is a Right Line.

For if it be not, draw the Right Line DEB in the Plane AB, from the Point D to the Point B, and the Right Line DFB in the Plane BC.

Then two Right Lines DEB, DFB, have the ſame Terms, and include a Space, which is * abſurd. Therefore DEB, DFB, are not Right Lines. In the ſame manner we demonſtrate, that no other Line drawn from the Point D to the Point B, is a Right Line, beſides DB, the common Section of the Planes AB, BC. *If*, therefore, *two Planes cut each other, their common Section will be a Right Line*; which was to be demonſtrated.

* *Axiom* 10.

PROPOSITION IV.

THEOREM.

If to two Right Lines, cutting one another, a third ſtands at Right Angles in the common Section, it ſhall be alſo at Right Angles to the Plane drawn thro' the ſaid Lines.

LET the Right Line EF ſtand at Right Angles to the two Right Lines AB, CD, in the common Section E. I ſay, EF is alſo at Right Angles to the Plane drawn thro' AB, CD.

For, take the Right Lines EA, EB, CE, DE,
equal; and thro' E any-how draw the Right Line
GEH; and join AD, CB; and from the Point F
let there be drawn FA, FG, FD, FC, FH, FB:
Then, becauſe two Right Lines AE, ED, are equal
* 15. 1. to two Right Lines CE, EB, and they contain * the
† 4. 1. equal Angles AED, CEB; the Baſe AD ſhall be†
equal to the Baſe CB, and the Triangle AED equal
to the Triangle CEB; and ſo likewiſe is the Angle
DAE equal to the Angle EBC; but the Angle AEG
is * equal to the Angle BEH, therefore AGE,
BEH, are two Triangles, having two Angles of the
one equal to the two Angles of the other, each to each,
and one Side AE equal to one Side EB, *viz.* thoſe
that are at the equal Angles; and ſo the other Sides
‡ 26. 1. of the one will be ‡ equal to the other Sides of the
other. Therefore GE is equal to EH, and AG to
BH; and ſince AE is equal to EB, and FE is common,
and at Right Angles, the Baſe AF ſhall be†
equal to the Baſe FB: For the ſame Reaſon likewiſe,
ſhall CF be equal to FD. Again, becauſe AD is
equal to CB, and AF to FB, the two Sides FA,
AD, will be equal to the two Sides FB, BC, each
to each; but the Baſe DF has been proved equal to
§ 8. 5. the Baſe FC. Therefore the Angle FAD is § equal
to the Angle FBC: Moreover, AG has been proved
equal to BH; but FB alſo is equal to AF. Therefore
the two Sides FA, AG, are equal to the two
Sides FB, BH; and the Angle FAG is equal to the
Angle FBH, as has been demonſtrated; wherefore
the Baſe GF is § equal to the Baſe FH. Again, becauſe
GE has been proved equal to EH, and EF is
common, the two Sides GE, EF, are equal to the
two Sides HE, EF; but the Baſe HF is equal to the
Baſe FG; therefore the Angle GEF is § equal to
the Angle HEF, and ſo both the Angles GEF,
HEF, are Right Angles: Therefore FE makes Right-Angles
with GH, which is any-how drawn thro' E.
After the ſame manner we demonſtrate, that FE is
at Right Angles to all Right Lines that are drawn in
* *Def.* 3. *of this.* the Plane to it; but a Right Line is * at Right Angles
to a Plane, when it is at Right Angles to all Right
Lines drawn to it in the Plane. Therefore FE is at
Right Angles to a Plane drawn thro' the Right Lines
AB,

AB, CD. Wherefore, *if to two Right Lines cutting one another, a third ſtands at Right Angles in the common Section, it ſhall be alſo at Right Angles to the Plane drawn thro' the ſaid Lines*; which was to be demonſtrated.

PROPOSITION V.

THEOREM.

If to three Right Lines, touching one another, a third ſtands at Right Angles in their common Section, thoſe three Right Lines ſhall be in one and the ſame Plane.

LET the Right Line AB ſtand at Right Angles in the Point of Contact B, to the three Right Lines BC, BD, BE. I ſay, BC, BD, BE, are in one and the ſame Plane.

For, if they are not, let BD, BE, be in one Plane, and BC above it; and let the Plane paſſing thro' AB, BC, be produced, and it will * make the common Section, with the other Plane, a ſtrait Line, which let be BF. Then three Right Lines AB, BC, BF, are in one Plane drawn thro' AB, BC; and ſince AB ſtands at Right Angles to BD and BE, it ſhall be † at Right Angles to a Plane drawn thro' BE, DB; and ſo AB ſhall make ‡ Right Angles with all Right Lines touching, it that are in the ſame Plane; but BF being in the ſaid Plane, touches it. Wherefore the Angle ABF is a Right Angle, but the Angle ABC (by the *Hyp.*) is alſo a Right Angle. Therefore the Angle ABF is equal to the Angle ABC, and they are both in the ſame Plane, which cannot be; and ſo the Right Line BC is not above the Plane paſſing thro' BE and BD. Wherefore the three Lines BC, BD, BE, are in one and the ſame Plane. Therefore, *if to three Right Lines, touching one another, a third ſtands at Right Angles in their common Section, thoſe three Right Lines ſhall be in one and the ſame Plane*; which was to be demonſtrated.

* 3 *of this.*
† 4 *of this.*
‡ *Def.* 3.

PROPOSITION VI.

THEOREM.

If two Right Lines be perpendicular to one and the ſame Plane, thoſe Right Lines are parallel to one another.

LET two Right Lines AB, CD, be perpendicular to one and the ſame Plane. I ſay, AB is parallel to CD.

For, let them meet the Plane in the Points B, D, and join the Right Line BD, to which let DE be drawn in the ſame Plane at Right Angles; make DE equal to AB, and join BE, AE, AD.

Then, becauſe AB is at Right Angles to the aforeſaid Plane, it ſhall be * at Right Angles to all Right Lines, touching it, drawn in the Plane; but AB touches BD, BE, which are in the ſaid Plane. Therefore each of the Angles ABD, ABE, is a Right Angle. So, for the ſame Reaſon likewiſe, is each of the Angles CDB, CDE, a Right Angle. Then, becauſe AB is equal to DE, and BD is common, the two Sides AB, BD, ſhall be equal to the two Sides ED, DB; but they contain Right Angles. Therefore the Baſe AD is † equal to the Baſe BE. Again, becauſe AB is equal to DE, and AD to BE, the two Sides AB, BE, are equal to the two Sides ED, DA; but AE, their Baſe, is common. Wherefore the Angle ABE is ‡ equal to the Angle EDA; but ABE is a Right Angle. Therefore EDA is alſo a Right Angle; and ſo ED it perpendicular to DA; but it is alſo perpendicular to BD and DC. Therefore ED is at Right Angles in the Point of Contact to three Right Lines BD, DA, DC. Wherefore theſe three laſt Right Lines are * in one Plane: But BD, DA, are in the ſame Plane as AB is; for every Triangle is † in the ſame Plane. Therefore it is neceſſary, that AB, BD, DC, be in one Plane; but both the Angles ABD, BDC, are Right Angles. Wherefore AB is ‡ parallel to CD. Therefore, *if two Right Lines be perpendicular to one and the ſame Plane, thoſe Right Lines are parallel to one another*; which was to be demonſtrated.

* *Def.* 3. *of this.*

† 4. 1.

‡ 8. 1.

* 5 *of this.*

† 2 *of this.*

‡ 28. 1.

PRO-

PROPOSITION VII.

THEOREM.

If there be two parallel Lines, and any Point be taken in both of them, the Right Lines joining thoſe Points ſhall be in the ſame Planes as the Parallels are.

LET AB, CD, be two parallel Right Lines, in which are taken any Points, E, F. I ſay, a Right Line joining the Points E, F, are in the ſame Plane as the Parallels are.

For if it be not, let it be elevated above the ſame, if poſſible, as EGF; thro' which let ſome Plane be drawn, whoſe Section, with the Plane in which the Parallels are, let * be the Right Line EF; then the two Right Lines EGF, EF, will include a Space, which is † abſurd. Therefore a Right Line drawn from the Point E to the Point F, is not elevated above the Plane, and conſequently it muſt be in that paſſing thro' the Parallels AB, CD. Wherefore, *if there be two parallel Lines, and any Points be taken in both of them, the Right Line joining theſe Points ſhall be in the ſame Plane as the Parallels are*; which was to be demonſtrated.

* 3 *of this.*

† *Axiom* 10. 1.

PROPOSITION VIII.

THEOREM.

If there be two parallel Right Lines, one of which is perpendicular to ſome Plane, then ſhall the other be perpendicular to the ſame Plane.

LET AB, CD, be two parallel Right Lines, one of which, as AB, is perpendicular to ſome Plane. I ſay, the other CD is alſo perpendicular to the ſame Plane.

See the Fig. of Prop. VI.

For, let AB, CD, meet the Plane in the Points B, D, and let BD be joined; then AB, CD, BD, are * in one Plane. Let DE be drawn in the Plane at Right Angles to BD, and make DE equal to AB,

* 7 *of this.*

and join BE, AE, AD. Then, ſince AB is perpendicular to the Plane, it will * be perpendicular to all Right Lines, touching it, drawn in the ſame Plane; therefore each of the Angles ABD, ABE, is a Right Angle. And ſince the Right Line BD falls on the Right Lines AB, CD, the Angles ABD, CDB, ſhall be † equal to two Right Angles. Therefore the Angle CDB is alſo a Right Angle, and ſo CD is perpendicular to DB. And ſince AB is equal to DE, and BD is common, the two Sides AB, BD, are equal to the two Sides ED, DB. But the Angle ABD is equal to the Angle EDB; for each of them is a Right Angle. Therefore the Baſe AD is ‡ equal to the Baſe BE. Again, ſince AB is equal to DE, and BE to AD, the two Sides AB, BE, ſhall be equal to the two Sides ED, DA, each to each; but the Baſe AE is common. Wherefore the Angle ABE is * equal to the Angle EDA; but the Angle ABE is a Right Angle. Therefore EDA is alſo a Right Angle, and ED is perpendicular to DA; but it is likewiſe perpendicular to DB: Therefore ED ſhall alſo be † perpendicular to the Plane paſſing thro' BD, DA, and likewiſe ſhall be ‡ at Right Angles to all Right Lines, drawn in the ſaid Plane, that touch it. But DC is in the Plane paſſing thro' BD, DA, becauſe AB, BD, are * in that Plane; and DC is † in the ſame Plane that AB and BD are in. Wherefore ED is at Right Angles to DC, and ſo CD is at Right Angles to DE, as alſo to DB. Therefore CD ſtands at Right Angles in the common Section D, to two Right Lines DE, DB, mutually cutting one another; and accordingly is at Right Angles to the Plane paſſing thro' DE, DB; *which was to be demonſtrated.*

* *Def.* 3.
† 29. 1.
‡ 4. 1.
* 8. 1.
† 4 *of this.*
‡ *Def.* 3.
* 2 *of this.*
† 7 *of this.*

PRO-

PROPOSITION IX.

THEOREM.

Right Lines that are parallel to the ſame Right Line, not being in the ſame Plane with it, are alſo parallel to each other.

LET both the Right Lines A B, C D, be parallel to the Right Line E F, not being in the ſame Plane with it. I ſay, A B is parallel to C D.

For, aſſume any Point G in E F, from which Point G, let G H be drawn at Right Angles to E F, in the Plane paſſing thro' E F, A B: Alſo, let G K be drawn at Right Angles to E F in the Plane paſſing thro' E F, C D: Then, becauſe E F is perpendicular to G H, and G K, the Line E F ſhall alſo be * at Right Angles to a Plane * 4 *of this.* paſſing thro' G H, G K; but E F is parallel to A B. Therefore A B is † alſo at Right Angles to the Plane † 8 *of this.* paſſing thro' H G K. For the ſame Reaſon, C D is alſo at Right Angles to the Plane paſſing thro' H G K; and therefore A B, and C D, will be both at Right Angles to the Plane paſſing thro' H G K. But if two Right Lines be at Right Angles to the ſame Plane, they ſhall be ‡ parallel to each other. Therefore A B is parallel ‡ 6 *of this.* to C D; *which was to be demonſtrated.*

PROPOSITION X.

THEOREM.

If two Right Lines, touching one another, be parallel to two other Right Lines, touching one another, but not in the ſame Plane, thoſe Right Lines contain equal Angles.

LET two Right Lines A B, B C, touching one another, be parallel to two Right Lines D E, E F, touching one another, but not in the ſame Plane. I ſay, the Angle A B C is equal to the Angle D E F.

For, take B A, B C, E D, E F, equal to one another, and join A D, C F, B E, A C, D F: Then, becauſe B A is equal and parallel to E D, the Right Line

* 30. 1. AD ſhall alſo be * equal and parallel to BE. For the ſame Reaſon, CF will be equal and parallel to BE; therefore AD, CF, are both equal and parallel to BE. But Right Lines that are parallel to the ſame Right Line, not being in the ſame Plane with it, will † 9 *of this.* be † parallel to each other. Therefore AD is parallel and equal to CF; but AC, DF, joins them; ‡ 33. 1. wherefore AC is ‡ equal and parallel to DF. And becauſe two Right Lines AB, BC, are equal to two Right Lines DE, EF, and the Baſe AC equal to * 8. 1. the Baſe DF, the Angle ABC will be * equal to the Angle DEF. Therefore, *if two Right Lines touching one another, be parallel to two other Right Lines, touching one another, but not in the ſame Plane, thoſe Right Lines contain equal Angles*; which was to be demonſtrated.

PROPOSITION XI.

PROBLEM.

From a Point given above a Plane, to draw a Right Line perpendicular to that Plane.

LET A be a Point given above the given Plane BH. It is required to draw a Right Line from the Point A, perpendicular to the Plane BH.

Let a Right Line BC be any-how drawn in the * 12. 1. Plane BH, and let AD be drawn * from the Point A perpendicular to BC; then, if AD be perpendicular to the Plane BH, the thing required is already done. But if not, let DE be drawn in the Plane from the Point D at Right Angles to BC; and let AF be drawn * from the Point A perpendicular to DE. Laſtly, thro' F draw GH parallel to BC.

Then, becauſe BC is perpendicular to both DA and † 4 *of this.* DE, BC will alſo be † perpendicular to a Plane paſſing thro' ED, DA. But GH is parallel to BC. And if there are two Right Lines parallel, one of which is at Right Angles to ſome Plane, then ſhall ‡ 8 *of this.* the other be ‡ at Right Angles to the ſame Plane. Wherefore GH is at Right Angles to the Plane paſſing * *Def.* 3. thro' ED, DA, and ſo is * perpendicular to all the Right Lines in the ſame Plane that touch it. But AF, which is in the Plane paſſing thro' E D and D A, doth touch it. Therefore G H is perpendicular to A F, and ſo A F is perpendicular to G H; but A F likewiſe is perpendicular to D E; therefore A F is perpendicular to both H G, D E. But if a Right Line ſtands at Right Angles to two Right Lines, in their common Section, that Line will be † at Right Angles to the Plane paſſing thro' theſe Lines. Therefore A F is perpendicular to the Plane drawn thro' E D, G H; that is, to the given Plane B H. Therefore A F is drawn from the given Point A, above the given Plane B H, perpendicular to the ſaid Plane; *which was to be done.* † 14 *of this.*

PROPOSITION XII.

PROBLEM.

To erect a Right Line perpendicular to a given Plane, from a Point given therein.

LET A be a given Point in a given Plane M N. It is required to draw a Right Line from the Point A, at Right Angles, to the Plane M N.

Let ſome Point B be ſuppoſed above the given Plane, from which let B C be drawn * perpendicular to the ſaid Plane; and let A D be drawn † from A parallel to B C. * 11 *of this.* † 31. 1.

Then, becauſe A D, C B, are two parallel Right Lines, one of which, *viz.* B C, is perpendicular to the Plane M N; the other A D ſhall be ‡ alſo perpendicular to the ſame Plane. Therefore *a Right Line is erected perpendicular to a given Plane, from a Point given therein*; which was to be done. ‡ 8 *of this.*

PROPOSITION XIII.

THEOREM.

Two Right Lines cannot be erected at Right Angles, to a given Plane, from a Point therein given.

FOR, if it is poſſible, let two Right Lines AB, AC, be erected perpendicular to a given Plane on the ſame Side, at a given Point A, in a given Plane.

Then let a Plane be drawn thro' BA, AC, cutting the given Plane thro' A in the Right Line* DAE; therefore the Right Lines AB, AC, DAE, are in one Plane. And becauſe CA is perpendicular to the given Plane, it ſhall alſo be † perpendicular to all Right Lines drawn in that Plane, and touching it; but DAE, being in the given Plane, touches it. Therefore the Angle CAE is a Right Angle. For the ſame Reaſon, BAE is alſo a Right Angle; wherefore the Angle CAE is equal to BAE, and they are both in one Plane, which is abſurd. Therefore *two Right Lines cannot be erected at Right Angles, to a given Plane, from a Point therein given*; which was to be demonſtrated.

* 3 *of this.*

† *Def.* 3.

PROPOSITION XVI.

THEOREM.

Thoſe Planes, to which the ſame Right Line is perpendicular, are parallel to each other.

LET the Right Line AB be perpendicular to each of the Planes CD, EF. I ſay, theſe Planes are parallel.

For if they be not, let them be produced till they meet each other, and let the Right Line GH be the common Section, in which take any Point K, and join AK, BK. Then, becauſe AB is perpendicular to the Plane EF, it ſhall alſo be perpendicular to the Right Line BK, being in the Plane EF produced. Wherefore the Angle ABK is a Right Angle. And for the ſame Reaſon, BAK is alſo a Right Angle.

And ſo the two Angles A B K, B A K, of the Triangle A B K, are equal to two Right Angles, which is * impoſſible. Therefore the Planes C D, E F, being produced, will not meet each other, and ſo are neceſſarily parallel. Therefore *thoſe Planes, to which the ſame Right Line is perpendicular, are parallel to each other*; which was to be demonſtrated. * 17. 1.

PROPOSITION XV.

THEOREM.

If two Right Lines, touching one another, be parallel to two Right Lines, touching one another, and not being in the ſame Plane with them, the Planes drawn thro' thoſe Right Lines are parallel to each other.

LET two Right Lines A B, B C, touching one another, be parallel to two Right Lines D E, E F, touching one another, but not in the ſame Plane with them; I ſay, the Planes paſſing thro' A B, B C, and D E, E F, being produced, will not meet each other.

For, let B G be drawn from the Point B, perpendicular to the Plane paſſing thro' D E, E F, meeting the ſame in the Point G; and thro' G let G H be drawn parallel to E D, and G K parallel to E F; then, becauſe B G is perpendicular to the Plane paſſing thro' D E, E F, it ſhall alſo make * Right Angles with all Right Lines that touch it, and are in the ſame Plane; but G H and G K, which are both in the ſame Plane, touch it. Therefore each of the Angles B G H, B G K, is a Right Angle. And ſince B A is parallel to G H, the Angles G B A, G B H, are † equal to the Right Angles: But B G H is a Right Angle; wherefore G B A ſhall alſo be a Right Angle, and ſo B G is perpendicular to B A. For the ſame Reaſon, G B is alſo perpendicular to B C. Therefore, ſince a Right Line B G ſtands at Right Angles to two Right Lines B A, B C, mutually cutting each other; B G ſhall alſo be ‡ at Right Angles to the Plane drawn thro' B A, B C. But it is perpendicular to the Plane drawn thro' D E, E F; therefore B G is perpendicular to both the Planes drawn thro' A B, B C, and D E,

* *Def.* 3.

† 29. 1.

‡ 4 *of this.*

DE, EF. But thoſe Planes to which the ſame Right Line is perpendicular, are * parallel. Therefore the Plane drawn thro' AB, BC, is parallel to the Plane drawn thro' DE, EF. Wherefore, *if two Right Lines, touching one another, be parallel to two Right Lines, touching one another, and not being in the ſame Plane with them, the Planes drawn thro' theſe Right Lines are parallel to each other.*

* 14 *of this.*

PROPOSITION XVI.

THEOREM.

If two parallel Planes are cut by any other Plane, their common Sections will be parallel.

LET two parallel Planes, AB, CD, be cut by any Plane EFHG, and let their common Sections be EF, GH. I ſay, EF is parallel to GH.

For, if it is not parallel, EF, GH, being produced, will meet each other either on the Side FH, or EG. Firſt, let them be produced on the Side FH, and meet in K; then, becauſe EFK is in the Plane AB, all Points taking in EFK will be in the ſame Plane. But K is one of the Points that is in EFK. Therefore K is in the ſame Plane AB. For the ſame Reaſon K is alſo in the Plane CD. Wherefore the Planes AB, CD, will meet each other. But they do not meet, ſince they are ſuppoſed parallel. Therefore the Right Lines EF, GH, will not meet on the Side FH. After the ſame manner it is proved, that they will not meet, if produced, on the Side EG. But Right Lines, that will neither way meet each other, are parallel; therefore EF is parallel to GH. *If*, therefore, *two parallel Planes are cut by any other Plane, their common Sections will be parallel*; which was to be demonſtrated.

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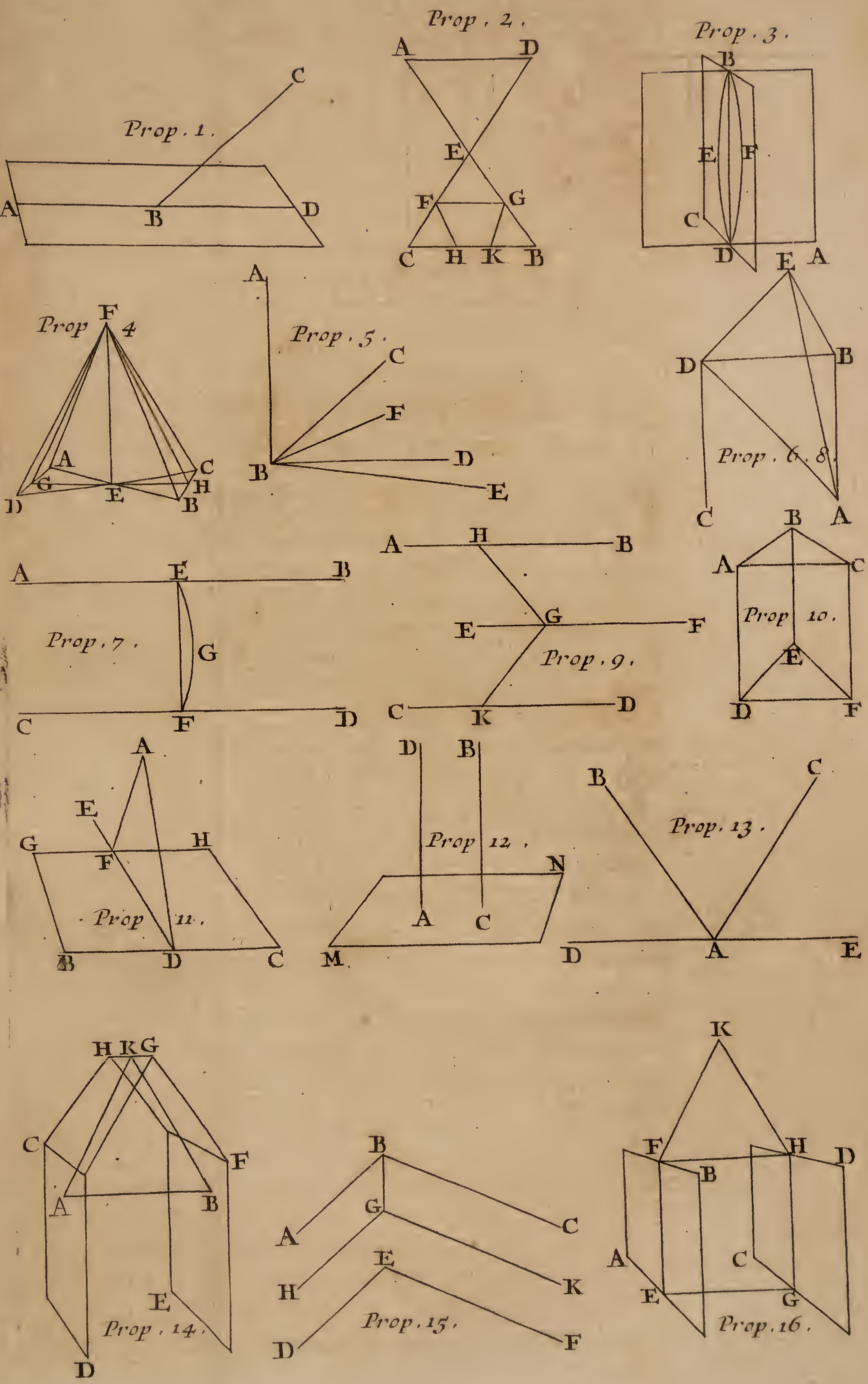


PROPOSITION XVII.

THEOREM.

If two Right Lines are cut by parallel Planes, they ſhall be cut in the ſame Proportion.

LET two Right Lines AB, CD, be cut by parallel Planes GH, KL, MN, in the Points A, E, B, C, F, D. I ſay, as the Right Line AE is to the Right Line EB, ſo is CF to FD.

For, let AC, BD, AD, be joined: Let AD meet the Plane KL in the Point X, and join EX, XF. Then, becauſe two parallel Planes KL, MN, are cut by the Plane EBDX, their common Sections EX, BD, are * parallel. For the ſame Reaſon, becauſe * 16 *of this.* two parallel Planes GH, KL, are cut by the Plane AXFC, their common Sections AC, FX, are parallel; and becauſe EX is drawn parallel to the Side BD of the Triangle ABD, it ſhall be as AE is to EB, ſo is † AX to XD. Again, becauſe XF is † 2. 6. drawn parallel to the Side AC of the Triangle ADC, it ſhall be † as AX to XD, ſo is CF to FD. But it has been proved, as AX is to XD, ſo is AE to EB. Therefore, as AE is to EB, ſo is ‡ CF to ‡ 11. 5. FD. Wherefore, *if two Right Lines are cut by parallel Planes, they ſhall be cut in the ſame Proportion*; which was to be demonſtrated.

PROPOSITION XVIII.

THEOREM.

If a Right Line be perpendicular to ſome Plane, then all Planes paſſing thro' that Line will be perpendicular to the ſame Plane.

LET the Right Line AB be perpendicular to the Plane CL. I ſay, all Planes that paſs thro' AB, are likewiſe perpendicular to the Plane CL.

For, let a Plane DE paſs thro' the Right Line AB, whoſe common Section, with the Plane CL, is the Right Line CE; and take ſome Point F in CE; from which let FG be drawn in the Plane DE, perpendicular to the Right Line CE. Then, becauſe AB is perpendicular to the Plane CL, it ſhall alſo be* perpendicular to the Right Lines which touch it, and are in the ſame Plane. Wherefore, it is perpendicular to CE; and conſequently, the Angle ABF is a Right Angle; but GFB is likewiſe a Right Angle. Therefore AB is parallel to FG. But AB is at Right Angles to the Plane CL. Therefore, FG will be† at Right Angles to that ſame Plane. But one Plane is perpendicular to another, when the Right Lines, drawn in one of the Planes perpendicular to the common Section of the Planes, are‡ perpendicular to the other Plane. But FG is drawn in one Plane DE, perpendicular to the common Section CE of the Planes. And it has been proved to be perpendicular to the Plane CL. Therefore the Plane DE is at Right Angles to the Plane CL. After the ſame manner it is demonſtrated, that all Planes, paſſing thro' the Right Line AB, are perpendicular to the Plane CL. Therefore, *if a Right Line be perpendicular to ſome Plane, then all Planes paſſing thro' that Line will be perpendicular to the ſame Plane*; which was to be demonſtrated.

* *Def.* 3.

† 8 *of this.*

‡ *Def.* 4 *of this.*

PROPOSITION XIX.

THEOREM.

If two Planes cutting each other, be perpendicular to ſome Plane, then their common Section will be perpendicular to that ſame Plain.

LET two Planes AB, BC, cutting each other, be perpendicular to ſome third Plane, and let their common Section be BD. I ſay, BD is perpendicular to the ſaid third Plane, which let be ADC.

For, if poſſible, let BD not be perpendicular to the third Plane; and from the Point D, let DE be drawn in the Plane AB, perpendicular to AD; and let DF be drawn in the Plane BC, perpendicular to CD; then, becauſe the Plane AB is perpendicular to the third Plane, and DE is drawn in the Plane AB, perpendicular to their common Section AD, DE ſhall be * perpendicular to the third Plane. In like * *Def.* 4. manner we prove, that DF alſo is perpendicular to the ſaid Plane. Wherefore two Right Lines ſtand at Right Angles, to this third Plane, on the ſame Side, at the ſame Point D; which is † abſurd. Therefore, † 13 *of this.* to this third Plane cannot be erected any Right Lines perpendicular at D, and on the ſame Side, except BD, the common Section of the Planes AB, BC. Wherefore DB is perpendicular to the third Plane. *If*, therefore, *two Planes, cutting each other, be perpendicular to ſome Plane, then their common Section will be perpendicular to that ſame Plane*; which was to be demonſtrated.

PROPOSITION XX.

THEOREM.

If a ſolid Angle be contained under three plian Angles, any two of them, howſoever taken, are greater than the third.

LET the ſolid Angle A be contained under three plain Angles BAC, CAD, DAB. I ſay, any two of the Angles BAC, CAD, DAB, are greater than the third, howſoever taken.

For, if the Angles BAC, CAD, DAB, be equal, it is evident, that any two, howſoever taken, are greater than the third. But if not, let BAC be the greater; and make * the Angle BAE, at the Point A, * 23. 1. with the Right Line AB, in a Plane paſſing thro' BA, AC, equal to the Angle DAB; make AE equal to AD; thro' E draw BEC, cutting the Right Lines AB, AC, in the Points B, C, and join DB, DC. Then, becauſe DA is equal to AE, and AB is common, the two Sides DA, AB, are equal to the two Sides AE, AB; but the Angle DAB is equal to the Angle BAE. Therefore the Baſe DB is † equal to † 4. 1. the Baſe BE. And ſince the two Sides DB, DC, are greater than BC, and DB has been proved equal to BE, the remaining Side DC ſhall be greater than the remaining Side EC; and ſince DA is equal to AE, and AC is common, and the Baſe DC greater than the Baſe EC, the Angle DAC ſhall be ‡ greater ‡ 25. 1.

than the Angle EAC. But from Conſtruction, the Angle DAB is equal to the Angle BAE. Wherefore the Angles DAB, DAC, are greater than the Angle BAC. After this manner we demonſtrate, if any two other Articles be taken, that they are greater than the third. Therefore, *if a ſolid Angle be contained under three plain Angles, any two of them, howſoever taken, are greater than the third*; which was to be demonſtrated.

PROPOSITION XXI.

THEOREM.

Every ſolid Angle is contained under plain Angles together, leſs than four Right ones.

LET A be a ſolid Angle, contained under plain Angles BAC, CAD, DAB. I ſay, the Angles BAC, CAD, DAB, are leſs than four Right Angles.

For, take any Points B, C, D, in each of the Lines AB, AC, AD; and join BC, CD, DB. Then, becauſe the ſolid Angle at B is contained under three plain Angles CBA, ABD, CBD, any two of theſe
*20 *of this.* are * greater than the third. Therefore the Angles CBA, ABD, are greater than the Angle CBD. For the ſame Reaſon, the Angles BCA, ACD, are greater than the Angle BCD; and the Angle CDA, ADB, greater than the Angle CDB. Wherefore the ſix Angles CBA, ABD, BCA, ACD, ADC, ADB, are greater than the three Angles CBD, BCD, CDB. But the three Angles CBD, BCD, CDB,
† 32. 1. are † equal to two Right Angles. Wherefore the ſix Angles CBA, ABD, BCA, ACD, ADC, ADB, are greater than two Right Angles. And ſince the three Angles of each of the Triangles ABC, ACD, ADB, are equal to two Right Angles, the nine Angles of thoſe Triangles CBA, BCA, BAC, ACD, CAD, ADC, ADB, ABD, DAB, are equal to ſix Right Angles. Six of which Angles CBA, BCA, ACD, ADC, ADB, ABD, are greater than two Right Angles. Therefore, the three other Angles BAC, CAD, DAB, which contain the ſolid Angle,

gle, will be leſs than four Right Angles. Wherefore *every ſolid Angle is contained under Angles together, leſs than four plain right ones*; which was to be demonſtrated.

PROPOSITION XXII.

Theorem.

If there be three plain Angles, whereof two, any how taken, are greater than the third, and the Right Lines that contain them be equal; then it is poſſible to make a Triangle of the Right Lines joining the equal Right Lines, which form the Angles.

LET ABC, DEF, GHK, be given plain Angles, any two whereof are greater than the third; and let the equal Right Lines AB, BC, DE, EF, GH, HK, contain them; and let AC, DF, GK, be joined. I ſay, it is poſſible to make a Triangle of AC, DF, GK; that is, any two of them, howſoever taken, are greater than the third.

For if the Angles at B, E, H, are equal, then AC, DF, GK, will be * equal, and any two of them greater than the third; but if not, let the Angles at B, E, H, be unequal; and let the Angle B be greater than either of the others at E or H. Then the Right Line AC will be † greater, than either DF or GK; and it is manifeſt, that AC, together with either DF, or GK, is greater than the other. I ſay likewiſe, that DF, GK, together, are greater than AC. For make ‡ at the Point B, with the Right Line AB, the Angle ABL equal to the Angle GHK; and make BL equal to either AB, BC, DE, EF, GH, HK, and join AL, CL. Then, becauſe the two Sides AB, BL, are equal to the two Sides GH, HK, each to each, and they contain equal Angles, the Baſe AL ſhall be equal to the Baſe GK. And ſince the Angles E and H are greater than the Angle ABC, whereof the Angle GHK is equal to the Angle ABL, the other Angle at E ſhall be greater than the Angle LBC. And ſince the two Sides LB, BC, are equal to the two Sides DE, EF, each to each, and the Angle

* 4. 1.
† 24. 1.
‡ 23. 1.

DEF is greater than the Angle LBC, the Baſe DF ſhall be * greater than the Baſe LC. But GK has been proved equal to AL. Therefore DF, GK, are greater then AL, LC; but AL, LC, are greater than AC. Wherefore DF, GK, ſhall be much greater than AC. Therefore any two of the Right Lines AC, DF, GK, howſoever taken, are greater than the other: And ſo a Triangle may be made of AC, DF, GK, *which was to be demonſtrated.*

* 24. 1.

PROPOSITION XXIII.

PROBLEM.

To make a ſolid Angle of three plain Angles, whereof any two, howſoever taken, are greater than the third; but theſe three Angles muſt be leſs than four Right Angles.

LET ABC, DEF, GHK, be three plain Angles given, whereof any two, howſoever taken, are greater than the other, and let the ſaid three Angles be leſs than four Right Angles. It is required to make a ſolid Angle of three plain Angles equal to ABC, DEF, GHK.

Let the Right Lines AB, BC, DE, EF, GH, HK, be cut off equal, and join AC, DF, GK; then it is poſſible to make * a Triangle of three Right Lines equal to AC, DF, GK: And ſo let † the Triangle LMN be made, ſo that AC be equal to LM, and DF to MN, and GK to LN; and let the Circle LMN be deſcribed ‡ about the Triangle, whoſe Centre let be X, which will be either within the Triangle LMN, or on one Side thereof, or without the ſame.

* 22 *of this.*
† 22. 1.
‡ 5. 4.

Firſt, let it be within, and join LX, MX, NX. I ſay, AB is greater than LX. For if this be not ſo, AB ſhall be either equal to LX, or leſs. Firſt, let it be equal; then, becauſe AB is equal to LX, and alſo to BC, LX ſhall be equal to BC; but LX is equal to XM. Therefore the two Sides AB, BC, are equal to the two Sides LX, XM, each to each; but the Baſe AC is put equal to the Baſe LM. Wherefore the Angle ABC ſhall be * equal to the Angle LXM.

* 8. 1.

LXM. For the ſame Reaſon, the Angle DEF is equal to the Angle MXN, and the Angle GHK to the Angle NXL. Therefore the three Angles ABC, DEF, GHK, are equal to the three Angles LXM, MXN, NXL. But the three Angles LXM, MXN, NXL, are * equal to four Right Angles: * *Cor.* 15. 1. And ſo the three Angles ABC, DEF, GHK, ſhall alſo be equal to four Right Angles; but they are put leſs than four Right Angles, which is abſurd. Therefore AB is not equal to LX, I ſay alſo, it is neither leſs than LX; for if this be poſſible, make XO equal to AB, and XP to BC, and join OP. Then, becauſe AB is equal to BC, XO ſhall be equal to XP; and the remaining Part OL equal to the remaining Part PM: And ſo LM is † parallel to OP, and the † 2. 6. Triangle LMX is equiangular to the Triangle OPX. Wherefore XL is ‡ to LM, as XO is to OP; and ‡ 4. 6. (by Alternation) as XL is to XO, ſo is LM to OP. But LX is greater than XO. Therefore LM ſhall alſo be greater than OP. But LM is put equal to AC. Wherefore AC ſhall be greater than OP. And ſo, becauſe the two Right Lines AB, BC, are equal to the two Right Lines OX, XP, and the Baſe AC greater than the Baſe OP; the Angle ABC will be * greater than the Angle OXP. In like * 25. 1. manner, we demonſtrate that the Angle DEF is greater than the Angle MXN, and the Angle GHK, than the Angle NXL. Therefore the three Angles ABC, DEF, GHK, are greater than the three Angles LXM, MXN, NXL. But the Angles ABC, DEF, GHK, are put leſs than four Right Angles. Therefore the Angles LXM, MXN, NXL, ſhall be leſs by much than four Right Angles, and alſo equal † to four Right Angles; which is abſurd. † *Cor.* 15. 1. Wherefore AB is not leſs than LX. It has alſo been prov'd not to be equal to it. Therefore it muſt neceſſarily be greater. On the Point X raiſe ‡ XR, ‡ 12 *of this.* perpendicular to the Plane of the Circle LMN; whoſe Length let be ſuch, that the Square thereof be equal to the Exceſs, by which the Square of AB exceeds the Square of LX; and let RL, RM, RN, be joined. Becauſe RX is perpendicular to the Plane of the Circle LMN, it ſhall alſo be * perpendicular * *Def.* 3. to LX, MX, NX. And becauſe LX is equal to

XM, and XR is common, and at Right Angles to
4. 1. them, the Baſe LR ſhall be equal to the Baſe RM.
For the ſame Reaſon, RN is equal to RL, or RM.
Therefore three Right Lines, RL, RM, RN, are
equal to each other. And becauſe the Square of XR
is equal to the Exceſs, by which the Square of AB
exceeds the Square of LX; the Square of AB will be
equal to the Squares of LX, XR together: But the
47. 1. Square of RL is † equal to the Squares of LX, XR:
For LXR is a Right Angle. Therefore the Square of
AB will be equal to the Square of RL; and ſo AB
is equal to RL. But BC, DE, EF, GH, HK,
are every of them equal to AB; and RN, or RM,
equal to RL. Wherefore AB, BC, DE, EF, GH,
HK, are each equal to RL, RM, or RN: And
ſince the two Sides RL, RM, are equal to the two
Sides AB, BC, and the Baſe LM is put equal to the
‡ 8. 1. Baſe AC, the Angle LRM ſhall be ‡ equal to the
Angle ABC. For the ſame Reaſon the Angle MRN
is equal to the Angle DEF, and the Angle LRN
equal to the Angle GHK. Therefore a ſolid Angle
is made at R of three plain Angles LRM, MRN,
LRN, equal to three plain Angles given, ABC,
DEF, GHK.

Now let the Centre of the Circle X be in one Side
of the Triangle, *viz.* in the Side MN, and join
XL. I ſay again, that AB is greater than LX. For,
if it be not ſo, AB will be either equal, or leſs than
LX. Firſt, let it be equal; then the two Sides AB,
BC, are equal to the two Sides MX, LX, that is,
they are equal to MN; but MN is put equal to DF.
* 20. 1. Therefore DE, EF, are equal to DF, which is * impoſſible.
Therefore AB is not equal to LX. In
like manner, we prove that it is neither leſſer; for
the Abſurdity will much more evidently follow.
Therefore AB is greater than LX. And if in like
manner, as before, the Square of RX be made equal
to the Exceſs, by which the Square of AB exceeds
the Square of LX, and RX be raiſed at Right Angles
to the Plane of the Circle, the Problem will be
done.

Laſtly, let the Centre X of the Circle be without
the Triangle LMN; and join LX, MX, NX.
I ſay, AB is greater than LX. For if it be not, it
muſt

muſt either be equal, or leſs. Firſt, let it be equal; then the two Sides AB, BC, are equal to the two Sides MX, XL, each to each; and the Baſe AC is equal to the Baſe ML; therefore the Angle ABC is equal to the Angle MXL. For the ſame Reaſon, the Angle GHK is equal to the Angle LXN; and ſo the whole Angle MXN is equal to the two Angles ABC, GHK; but the Angles ABC, GHK, are greater than the Angle DEF. Therefore the Angle MXN is greater than DEF; but becauſe the two Sides DE, EF, are equal to the two Sides MX, XN; and the Baſe DF is equal to the Baſe MN, the Angle MXN ſhall be equal to the Angle DEF; but it has been proved greater, which is abſurd. Therefore AB is not equal to LX. Moreover we will prove, that it is not leſs; wherefore it ſhall be neceſſarily greater. And if, again, XR be raiſed at Right Angles to the Plane of the Circle, and made equal to the Side of that Square, by which the Square of AB exceeds the Square of LX, the Problem will be determined. Now, I ſay, AB is not leſs than LX; for if it is poſſible, that it can be leſs, make XO equal to AB, and XP equal to BC, and join OP. Then, becauſe AB is equal to BC, XO ſhall be equal to XP, and the remaining Part OL equal to the remaining Part PM; Therefore LM is * parallel to * 2. 6. PO, and the Triangle LMX equiangular to the Triangle PXO. Wherefore, as † XL is to LM, ſo † 4. 6. is XO to OP: And (by Alternation) as LX is to XO, ſo is LM to OP; but LX is greater than XO; therefore LM is greater than OP; but LM is equal to AC; wherefore AC ſhall be greater than OP. And ſo, becauſe the two Sides AB, BC, are equal to the two Sides OX, XP, each to each; and the Baſe AC is greater than the Baſe OP; the Angle ABC ſhall be ‡ greater than the Angle OXP. So likewiſe, ‡ 25. 1. if XR be taken equal to XO or XP, and OR be joined, we prove that the Angle GHK is greater than the Angle OXR. At the Point X, with the Right Line LX, make the Angle LXS equal to the Angle ABC, and the Angle LXT equal to the Angle GHK, and XS, XT, each equal to XO; and join OS, OT, ST. Then, becauſe the two Sides AB, BC, are equal to the two Sides OX, XS, and the

the Angle ABC is equal to the Angle OXS, the Baſe AC; that is, LM ſhall be equal to the Baſe OS. For the ſame Reaſon, LN is alſo equal to OT. And ſince the two Sides ML, LN, are equal to the two Sides OS, OT, and the Angle MLN greater than the Angle SOT; the Baſe MN ſhall be greater than the Baſe ST; but MN is equal to DF; therefore DF ſhall be greater than ST. Wherefore, becauſe the two Sides DE, EF, are equal to the two Sides SX, XT, and the Baſe DF is greater than the Baſe ST, the Angle DEF ſhall be greater than the Angle SXT; but the Angle SXT is equal to the Angles ABC, GHK. Therefore the Angle DEF is greater than the Angles ABC, GHK; but it is alſo leſs, which is abſurd; *which was to be demonſtrated.*

PROPOSITION XXIV.

THEOREM.

If a Solid be contained under ſix parallel Planes, the oppoſite Planes thereof, are equal Parallelograms.

LET the Solid CDGH be contained under parallel Planes AC, GF, BG, CE, FB, AE. I ſay, the oppoſite Planes thereof are equal Parallelograms.

For, becauſe the parallel Planes BG, CE, are cut by the Plane AC, their common Sections are* parallel; wherefore AB is parallel to CD. Again, becauſe the two parallel Planes BF, AE, are cut by the Plane AC, their common Sections are parallel; therefore AD is parallel to BC; but AB has been proved to be parallel to CD; wherefore AC ſhall be a Parallelogram. After the ſame manner, we demonſtrate that CE, FG, GB, BF, or AE, is a Parallelogram.

* 16 *of this.*

Let AH, DF, be joined. Then, becauſe AB is parallel to DC, and BH to CF, the Lines AB, BH, touching each other, ſhall be parallel to the Lines DC, CF; touching each other, and not being in the ſame Plane; wherefore they ſhall † contain equal Angles. And ſo the Angle ABH is equal to the Angle DCF.

† 10 *of this.*

DCF. And ſince the two Sides AB, BH, are † equal ‡ 34. 1.
to the two Sides DC, CF, and the Angle ABH equal
to the Angle DCF, the Baſe AH ſhall be * equal to * 4. 1.
the Baſe DF, and the Triangle ABH equal to the
Triangle DCF. And ſince the Parallelogram BG is
† double to the Triangle ABH, and the Parallelogram † 41. 1.
CE, to the Triangle DCF, the Parallelogram BG
ſhall be equal to the Parallelogram CE. In like manner, we demonſtrate that the Parallelogram AC is equal to the Parallelogram GF, and the Parallelogram AE equal to the Parallelogram BF. *If*, therefore, *a Solid be contained under ſix parallel Planes, the oppoſite Planes thereof are equal Parallelograms*; which was to be demonſtrated.

Coroll. It follows from what has been now demonſtrated, that if a Solid be contained under ſix parallel Planes, the oppoſite Planes thereof are ſimilar and equal, becauſe each of the Angles are equal, and the Sides about the equal Angles are proportional.

PROPOSITION XXV.

Theorem.

If a ſolid Parallelopipedon be cut by a Plane, parallel to oppoſite Planes; then as Baſe is to Baſe, ſo ſhall Solid be to Solid.

LET the ſolid Parallelopipedon ABCD be cut by a Plane YE, parallel to the oppoſite Planes RA, DH. I ſay, as the Baſe EFΦA is to the Baſe EHCF, ſo is the Solid ABFY to the Solid EGCD.

For, let AH be both ways produced, and make
HM, MN, *&c.* equal to EH, and AK, KL, *&c.*
equal to AE; and let the Parallelograms LO, KΦ,
HX, MS, as likewiſe the Solids LP, KR, HΩ,
MT, be completed. Then, becauſe the Right Lines
LK, KA, AE, are equal, the Parallelograms LO,
KΦ, AF, ſhall be * alſo equal; as likewiſe the Pa- * 1. 6.
rallelograms KΞ, KB, AG: And moreover † the * 24 *of this.*
Parallelograms LΨ, KP, AR; for they are oppoſite.
For the ſame Reaſon, the Parallelograms EC, HX,
MS, alſo are equal to each other; as alſo the Parallelograms

lelograms HG, HI, IN; and ſo are the Parallelograms DH, MΩ, NT. Therefore three Planes of the Solid LP are equal to three Planes of the Solid KR, or AY, each to each; and the Planes oppoſite to theſe, are equal to them. Therefore the three Solids LP, KR, AY, will be equal ‡ to each other. For the ſame Reaſon, the three Solids ED, HΩ, MT, are equal to each other. Therefore the Baſe LF is the ſame Multiple of the Baſe AF, as the Solid LY is of the Solid AY. For the ſame Reaſon, the Baſe NF is the ſame Multiple of the Baſe HF, as the Solid NY is of the Solid ED: And if the Baſe LF be equal to the Baſe NF, the Solid LY ſhall be equal to the Solid NY; and if the Baſe LF exceeds the Baſe NF, the Solid LY ſhall exceed the Solid NY; and if it be leſs, leſs. Wherefore, becauſe there are four Magnitudes, *viz.* the two Baſes AF, FH, and the two Solids AY, ED, whoſe Equimultiples are taken, to wit, the Baſe LF, and the Solid LY; and the Baſe NF, and the Solid NY: And ſince it is proved, if the Baſe LF exceeds the Baſe NF, then the Solid LY will exceed the Solid NY; if equal, equal; and leſs, leſs: Therefore, as the Baſe AF is to the Baſe FH, ſo is * the Solid AY to the Solid ED. Wherefore, *if a ſolid Parallelopipedon be cut by a Plane, parallel to oppoſite Planes; then as Baſe is to Baſe, ſo ſhall Solid be to Solid;* which was to be demonſtrated.

‡ Def. 10. of this.

* Def. 6. 5.

PROPOSITION XXVI.

THEOREM.

At a Right Line given, and at a Point given in it, to make a ſolid Angle equal to a ſolid Angle given.

LET AB be a Right Line given, A a given Point in it, and D a given ſolid Angle contained under the plain Angles EDC, EDF, FDC; it is required to make a ſolid Angle at the given Point A, in the given Right Line AB, equal to the given ſolid Angle D.

Aſſume any Point F in the Right Line DF, from which let FG be drawn * perpendicular to the Plane paſſing

* 11 of this.

paſſing thro' E D, D C, meeting the ſaid Plane in the Point G; and join D G; make † the Angles B A L, † 23. 1. B A K, at the given Point A, with the Right Line A B, equal to the Angles E D C, E D G.

Laſtly, make A K equal to D G; and at the Point K erect ‡ H K at Right Angles to the Plane paſſing ‡ 12 *of this.* thro' B A L; and make K H equal to G F, and join H A. I ſay, the ſolid Angle at A, which is contained under the three plain Angles B A L, B A H, H A L, is equal to the ſolid Angle at D, which is contained under the plain Angles E D C, E D F, F D C: For let the equal Right Lines A B, D E, be taken; and join H B, K B, F E, G E. Then, becauſe F G is perpendicular to the Plane paſſing through E D, D C, it ſhall be * perpendicular to all the Right Lines touching * *Def.* 3. *of this.* it, that are in the ſaid Plane. Wherefore both the Angles F G D, F G E, are Right Angles. For the ſame Reaſon, both the Angles H K A, H K B, are Right Angles; and becauſe the two Sides K A, A B, are equal to the two Sides G D, D E, each to each, and contain equal Angles, the Baſe B K ſhall be ‡ equal to the † 4. 1. Baſe E G; but K H is alſo equal to G F, and they contain Right Angles; therefore H B ſhall be † equal to F E. Again, becauſe the two Sides A K, K H, are equal to the two Sides D G, G F, and they contain Right Angles, the Baſe A H ſhall be equal to the Baſe D F; but A B is equal to D E. Therefore the two Sides H A, A B, are equal to the two Sides F D, D E; but the Baſe H B is equal to the Baſe F E, and ſo the Angle B A H will be ‡ equal to the Angle ‡ 8. 1. E D F. For the ſame Reaſon, the Angle H A L is equal to the Angle F D C; for ſince, if A L be taken equal to D C, and K L, H L, G C, F C, be joined, the whole Angle B A L is equal to the whole Angle E D C; and the Angle B A C, a Part of the one, is put equal to the Angle E D G, a Part of the other; the Angle K A L remaining, will be equal to the Angle G D C remaining. And becauſe the two Sides K A, A L, are equal to the two Sides G D, D C, and they contain equal Angles, the Baſe K L will be equal to the Baſe G C; but K H is equal to G F; wherefore the two Sides L K, K H, are equal to the two Sides C G, G F; but they contain Right Angles; therefore the Baſe H L will be equal to the Baſe F C.

Again, becauſe the two Sides HA, AL, are equal to the two Sides FD, DC, and the Baſe HL is equal to the Baſe FC, the Angle HAL will be equal to the Angle FDC; but the Angle BAL is equal to the Angle EDC; *which was to be done.*

PROPOSITION XXVII.

PROBLEM.

Upon a Right Line given, to deſcribe a Parallelopipedon ſimilar, and in like manner ſituate to a ſolid Parallelopipedon given.

LET AB be a Right Line, and CD a given ſolid Parallelopipedon. It is required to deſcribe a ſolid Parallelopipedon upon the given Right Line AB, ſimilar and alike ſituate to the given ſolid Parallelepipedon CD.

Make a ſolid Angle at the given Point A, in the Right Line AB, which * is contained under the Angles BAH, HAK, KAB; ſo that the Angle BAH be equal to the Angle ECF, the Angle BAK to the Angle ECG, and the Angle HAK to the Angle GCF; and make as EC is to CG, ſo BA † to AK; and GC to CF, as KA to AH. Then (by Equality of Proportion) as EC is to CF, ſo ſhall BA be to AH; complete the Parallelogram BH, and the Solid AL. Then, becauſe it is as EC is to CG, ſo is BA to AK, *viz.* the Sides about the equal Angles ECG, BAK, proportional; the Parallelogram KB ſhall be ſimilar to the Parallelogram GE. Alſo, for the ſame Reaſon, the Parallelogram KH ſhall be ſimilar to the Parallelogram GF, and the Parallelogram HB to the Parallelogram FE. Therefore three Parallelograms of the Solid AL are ſimilar to three Parallelograms of the Solid CD; but theſe three Parallelograms are ‡ equal and ſimilar to their three oppoſite ones. Therefore the whole Solid AL will be ſimilar to the whole Solid CD; and ſo a ſolid Parallelopipedon AL is deſcribed upon the given Right Line AB, ſimilar and alike ſituate to tha given ſolid Parallelopipedon CD; *which was to be done.*

* 26 *of this.*

† 12. 6.

‡ Cor. 24. *of this.*

PRO-

PROPOSITION XXVIII.

THEOREM.

If a ſolid Parallelopipedon be cut by a Plane paſſing thro' the Diagonals of two oppoſite Planes, that Solid will be biſected by the Plane.

LET the ſolid Parallelopipedon AB be cut by the Plane CDEF, paſſing thro' the Diagonals CF, DE, of two oppoſite Planes. I ſay, the Solid AB is biſected by the Plane CDEF.

For, becauſe the Triangle CGF is * equal to the * 34. 1.
Triangle CBF, and the Triangle ADE to the Tri-
angle DEH, and the Parallelogram CA to † the † 24 *of this.*
Parallelogram BE, for it is oppoſite to it; and the
Parallelogram GE to the Parallelogram CH; the
Priſm contained by the two Triangles CGF, ADE,
and the three Parallelograms GE, AC, CE, is equal
to the Priſm contained under the two Triangles CFB,
DEH, and the three Parallelograms CH, BE, CE;
for they are contained under Planes equal in Number and Magnitude. Therefore the whole Solid AB
is biſected by the Plane CDEF; *which was to be demonſtrated.*

PROPOSITION XXIX.

THEOREM.

Solid Parallelepipedons, being conſtituted upon the ſame Baſe, and having the ſame Altitude, and whoſe inſiſtent Lines are in the ſame Right Lines, are equal to one another.

LET the ſolid Parallelopipedons CM, CN, be conſtituted upon the ſame Baſe AB, with the ſame Altitude, whoſe inſiſtent Lines AF, AG, LM, LN, CD, CE, BH, BK, are in the ſame Right Lines FN, DK. I ſay, the Solid CM is equal to the Solid CN.

For, becauſe CH, CK, are both Parallelograms,
CB ſhall be * equal to DH, or EK; wherefore DH * 34. 1.

is equal to EK. Let EH, which is common, be taken away, then the Remainder DE will be equal to the Remainder HK; and ſo the Triangle DEC is † 8. 1. † equal to the Triangle HKB, and the Parallelogram DG equal to the Parallelogram HN. For the ſame Reaſon, the Triangle AFG is equal to the Triangle ‡ 24 of this. LMN; and the Parallelogram CF ‡ to the Parallelogram BM, and the Parallelogram CG to the Parallelogram BN; for they are oppoſite. Therefore the Priſm contained under the two Triangles AFG, DEC, and the three Parallelograms AD, DG, GC, is equal to the Priſm contained under the two Triangles LMN, HBK, and the three Parallelograms BM, NH, BN. Let the common Solid, whoſe Baſe is the Parallelogram AB, oppoſite to the Parallelogram GEHM, be added; then the whole ſolid Parallelopipedon CM is equal to the whole ſolid Parallelopipedon CN. Therefore *ſolid Parallelopipedons, being conſtituted upon the ſame Baſe, and having the ſame Altitude, and whoſe inſiſtent Lines are in the ſame Right Lines, are equal to one another*; which was to be demonſtrated.

PROPOSITION XXX.

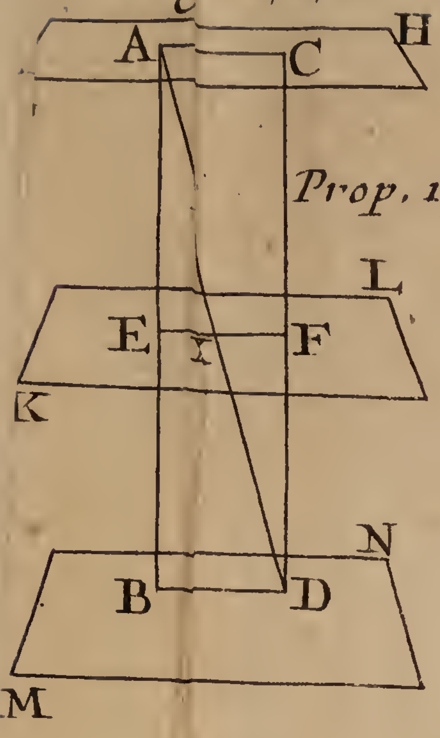
THEOREM.

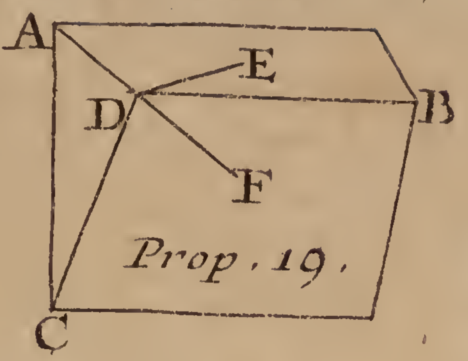
Solid Parallelopipedons, being conſtituted upon the ſame Baſe, and having the ſame Altitude, whoſe inſiſtent Lines are not placed in the ſame Right Lines, are equal to one another.

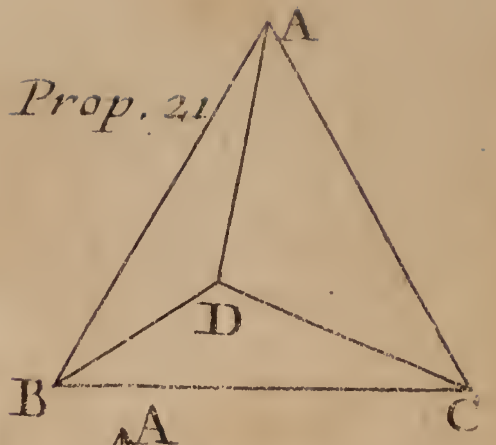
LET there be ſolid Parallelopipedons CM, CN, having equal Altitudes, and ſtanding on the ſame Baſe AB, and whoſe inſiſtent Lines AF, AG, LM, LN, CD, CE, BH, BK, are not in the ſame Right Lines. I ſay, the Solid CM is equal to the Solid CN.

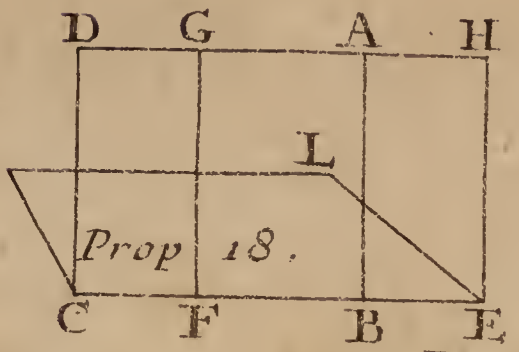
For, let NK, DH, and GE, FM, be produced, meeting each other in the Points R, X; let alſo FM, GE, be produced to the Points O, P, and join AX, LO, CP, BR. The Solid CM, whoſe Baſe is the Parallelogram ACBL, being oppoſite to the Parallelogram FDHM, is * equal to the Solid CO, * 29 of this. whoſe

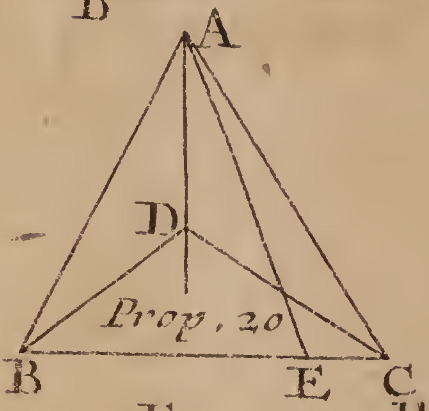
Book 11. Plate 2.

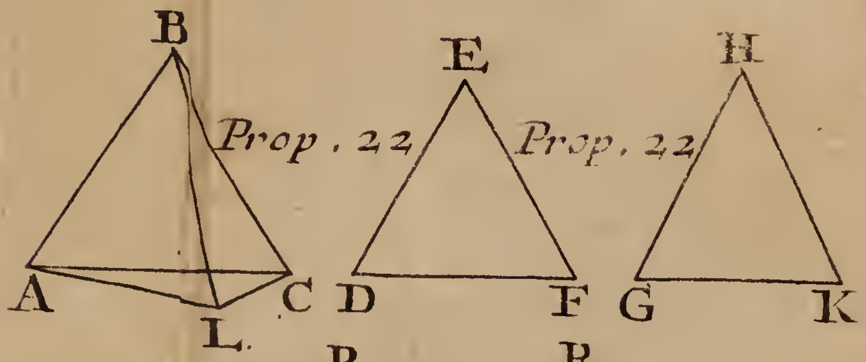


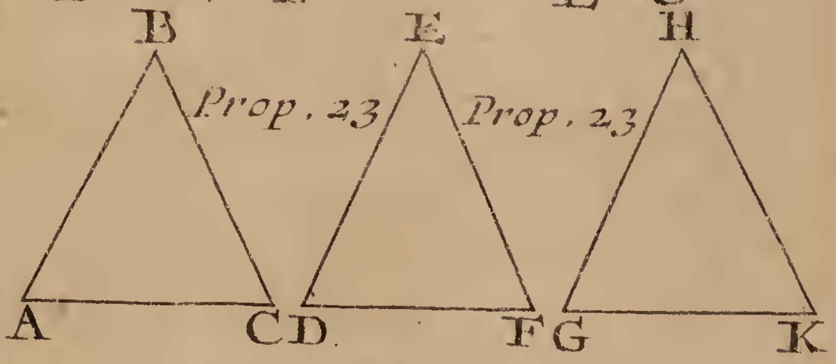


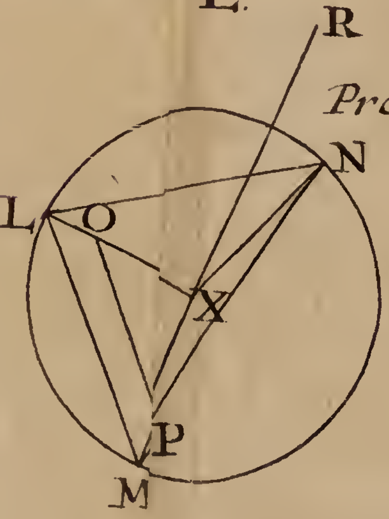


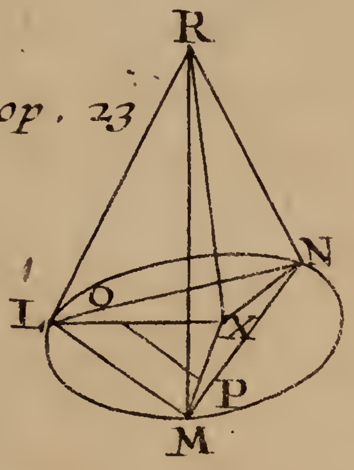


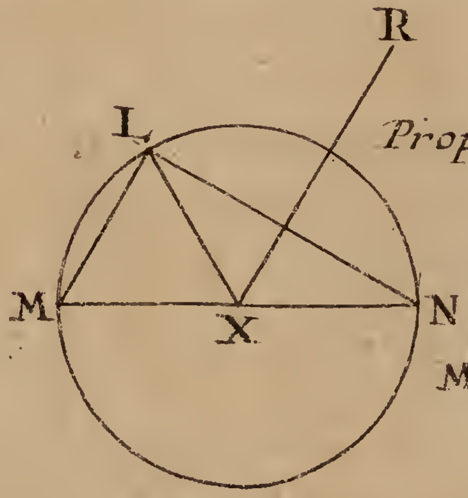


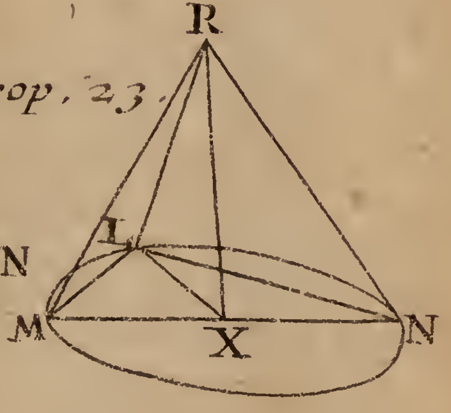


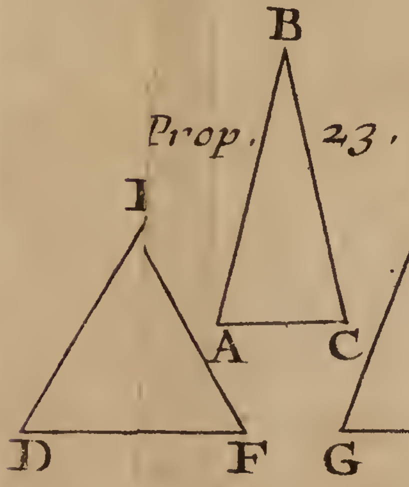


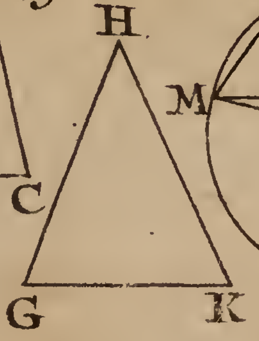


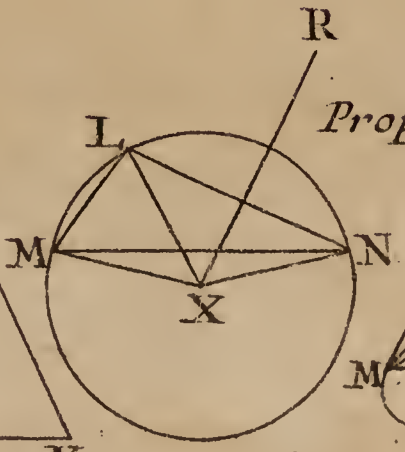


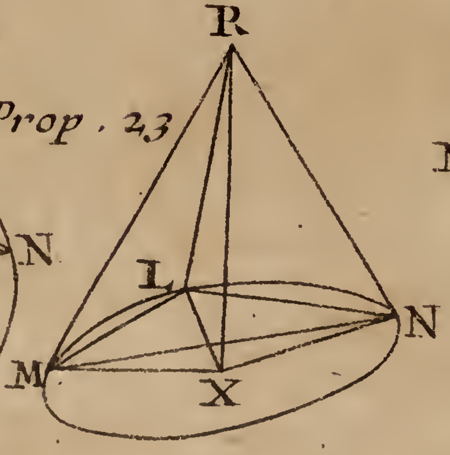


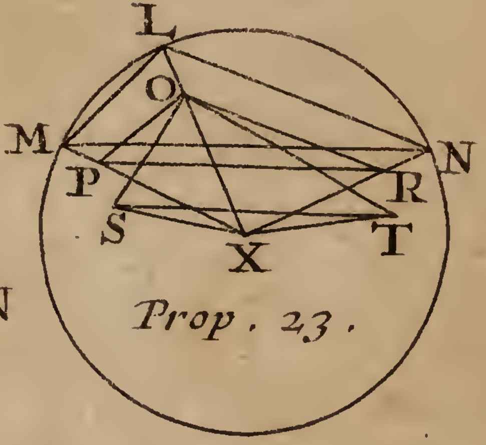


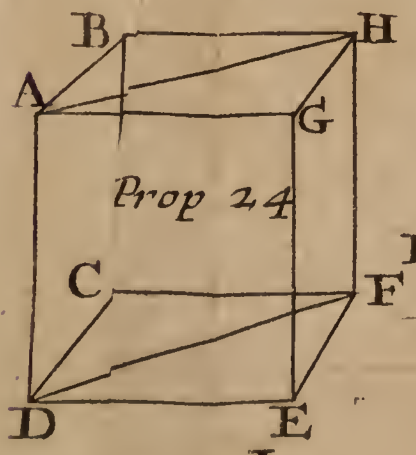


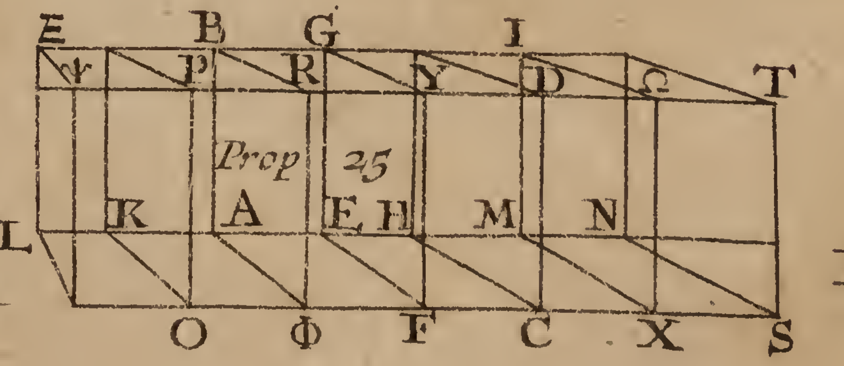


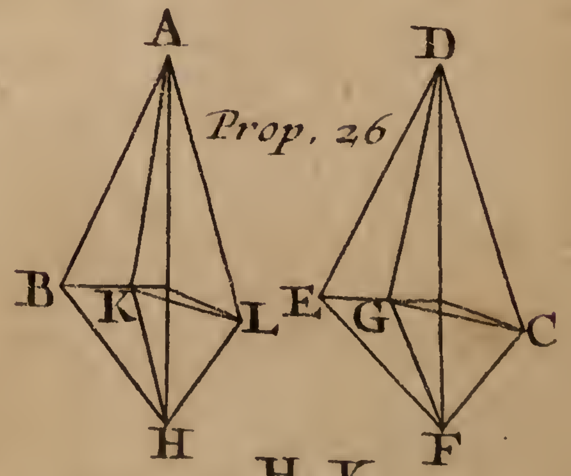


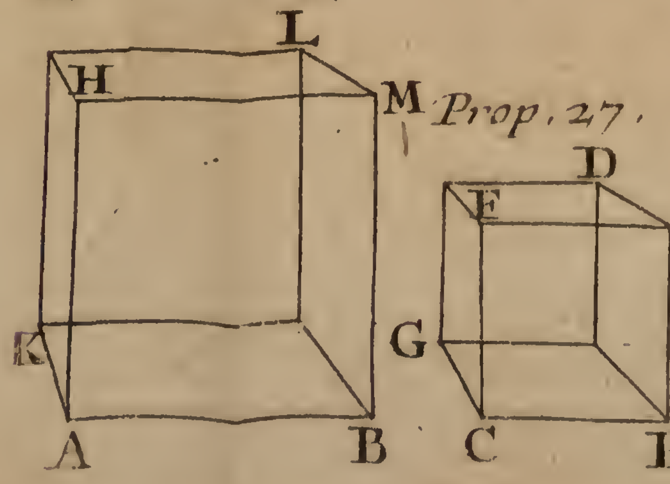


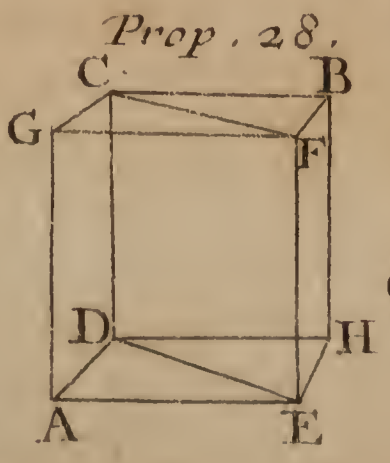


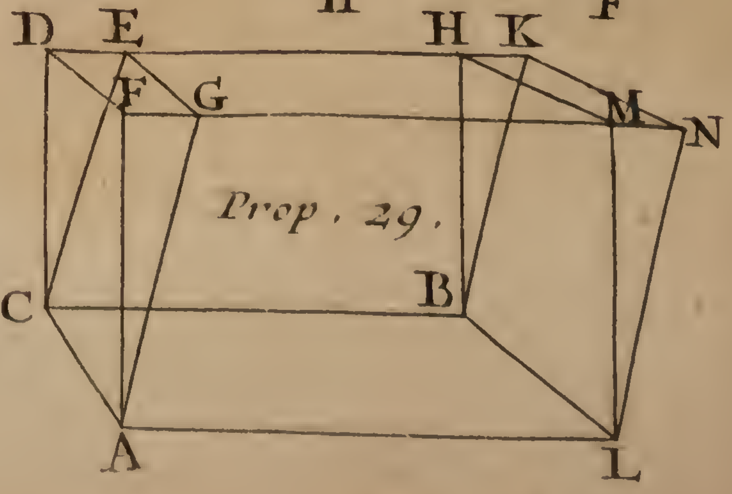












whoſe Baſe is the Parallelogram ACBL, being oppoſite to XPRO; for they ſtand upon the ſame Baſe ACBL; and the inſiſtent Lines AF, AX, LM, LO, CD, CP, BH, BR, are in the ſame Right Lines, FO, DR; but the Solid CO, whoſe Baſe is the Parallelogram ACBL, being oppoſite to XPRO, is * equal to the Solid CN, whoſe Baſe is the Parallelogram ACBL, being oppoſite to GEKN; for they ſtand upon the ſame Baſe ACBL, and their inſiſtent Lines AG, AX, CE, CP, LN, LO, BK, BR, are in the ſame Right Lines GP, NR; wherefore the Solid CM ſhall be equal to the Solid CN. Therefore *ſolid Parallelopipedons, being conſtituted upon the ſame Baſe, and having the ſame Altitude, whoſe inſiſtent Lines are not placed in the ſame Right Lines, are equal to one another*; which was to be demonſtrated.

* *29 of this.*

PROPOSITION XXXI.

THEOREM.

Solid Parallelopipedons, being conſtituted upon equal Baſes, and having the ſame Altitude, are equal to one another.

LET AE, CF, be ſolid Parallelopipedons conſtituted upon the equal Baſes AB, CD, and having the ſame Altitude. I ſay, the Solid AE is equal to the Solid CF.

Firſt, let HK, BE, AG, LM, OP, DF, CΞ, RS, be at Right Angles to the Baſes AB, CD; let the Angle ALB not be equal to the Angle CRD, and produce CR to T, ſo that RT be equal to AL: Then make the Angle TRY, at the Point R, in the Right Line RT, equal * to the Angle ALB; make RY equal to LB; draw XY thro' the Point Y parallel to RT, and complete the Parallelogram RX, and the Solid ΨY. Therefore, becauſe the two Sides TR, RY, are equal to the two Sides AL, LB, and they contain equal Angles, the Parallelogram RX ſhall be equal and ſimilar to the Parallelogram HL. And again, becauſe AL is equal to RT, and LM to RS, and they contain equal Angles, the Parallelogram

* 23. 1.

lelogram RΨ ſhall be equal and ſimilar to the Parallelogram AM. For the ſame Reaſon, the Parallelogram LE is equal and ſimilar to the Parallelogram SY. Therefore three Parallelograms of the Solid AE are equal and ſimilar to three Parallelograms of the Solid ΨY; and ſo the three oppoſite ones of one † *24 of this.* Solid are † alſo equal and ſimilar to the three oppoſite ones of the other. Therefore the whole ſolid Paralleloppedon AE is equal to the whole ſolid Parallelopipidon ΨY. Let DR, XY, be produced, and meet each other in the Point Ω, and let TQ be drawn thro' T parallel to DΩ, and produce TQ, OD, till they meet in V, and complete the Solid ΩΨRI: Then the Solid ΨΩ, whoſe Baſe is the Pa- ‡ *29 of this.* rallelogram RΨ, and ΩΓ is that oppoſite to it, is ‡ equal to the Solid ΨY, whoſe Baſe is the Parallelogram RΨ, and YΦ is that oppoſite to it; for they ſtand upon the ſame Baſe RΨ, have the ſame Altitude, and their inſiſtent Lines RΩ, RY, TQ, TX, SZ, SN, ΨΓ, ΨΦ, are in the ſame Right Lines ΩX, ZΦ: But the Solid ΨY is equal to the Solid AE; and ſo AE is equal to the Solid ΨΩ. Again, becauſe the Parallelogram RYXT is equal to the Parollelogram ΩT, for it ſtands on the ſame Baſe RT, and between the ſame Parallels RT, ΩX; and the Parallelogram RYXT is equal to the Parallelogram CD, becauſe it is alſo equal to AB; the Parallelogram ΩT is equal to the Parallelogram CD, and DT is ſome other Parallelogram. Therefore, as the Baſe CD is to the Baſe DT, ſo is ΩT to TD; and becauſe the ſolid Parallelopipedon CI is cut by the Plane RF, being parallel to two oppoſite Planes, it * *25 of this.* ſhall be * as the Baſe CD is to the Baſe DT, ſo is the Solid CF to the Solid RI. For the ſame Reaſon, becauſe the ſolid Parallelopipedon ΩI is cut by the Plane RΨ parallel to two oppoſite Planes; as the Baſe ΩT is to the Baſe DT, ſo ſhall * the Solid ΩΨ be to the Solid RI; but as the Baſe CD is to the Baſe DT, ſo is the Baſe ΩT to TD. Therefore, as the Solid CF is to the Solid RI, ſo is the Solid ΩΨ to the Solid RI; and ſince each of the Solids CF, ΩΨ, has the ſame Proportion to the Solid RI, the Solid CF is equal to the Solid ΩΨ; but the Solid ΩΨ has been proved equal to the Solid AE; therefore

fore the Solid AE ſhall be † equal to the Solid CF. † 9. 5.

But, now let the inſiſtent Lines AG, HK, BE, LM, CN, OP, DF, RS, not be at Right Angles to the Baſes AB, CD. I ſay, again, that the Solid AE is equal to the Solid CF. Let there be drawn from the Points K, E, G, M, P, F, N, S, to the Plane wherein are the Baſes AB, CD, the Perpendiculars KΞ, ET, GY, MΦ, SI, FΨ, NΩ, PX, meeting the Plane in the Points Ξ, T, Y, Φ, I, Ψ, Ω, X; and join ΞT, YΦ, ΞY, TΦ, XΨ, XΩ, ΩI, ΨI; then the Solid KΦ is equal to the Solid PI; for they ſtand on equal Baſes KM, PS, have the ſame Altitude, and the inſiſtent Lines are at Right Angles to the Baſes; but the Solid KΦ is equal to the Solid AE, and the Solid PI to ‡ the Solid CF, ſince they ſtand upon the ‡ 29 *of this.* ſame Baſe, have the ſame Altitude, and their inſiſtent Lines are in the ſame Right Lines. Therefore the Solid AE ſhall be equal to the Solid CF. Wherefore *ſolid Parallelopipedons, being conſtituted upon equal Baſes, and having the ſame Altitude, are equal to one another*; which was to be demonſtrated.

PROPOSITION XXXII.

THEOREM.

Solid Parallelopipedons, that have the ſame Altitude, are to each other as their Baſes.

LET AB, CD, be ſolid Parallelopipedons, that have the ſame Altitude. I ſay, they are to one another as their Baſes; that is, as the Baſe AE is to the Baſe CF, ſo is the Solid AB to the Solid CD.

For, apply a Parallelogram FH to the Right Line FG, equal to the Parallelogram AE, and complete the ſolid Parallelopipedon GK upon the Baſe FH, having the ſame Altitude as CD has. Then the Solid AB is * equal to the Solid GK; for they ſtand * 31 *of this.* upon equal Baſes AE, FH, and have the ſame Altitude; and ſo, becauſe the ſolid Parallelopipedon CK is cut by the Plane DG, parallel to two oppoſite Planes, it ſhall be † as the Baſe HF is to the Baſe † 25 *of this.* FC, ſo is the Solid HD to the Solid DC; but the

Baſe FH is equal to the Baſe AE, and the Solid AB to the Solid FK. Therefore, as the Baſe AE is to the Baſe CF, ſo is the Solid AB to the Solid CD. Wherefore *ſolid Parallelopipedons, that have the ſame Altitude, are to each other as their Baſes*; which was to be demonſtrated.

PROPOSITION XXXIII.

THEOREM.

Similar ſolid Parallelopipedons are to one another in the triplicate Proportion of their homologous Sides.

LET AB, CD, be ſolid Parallelopipedons, and let the Side AE be homologous to the Side CF. I ſay, the Solid AB, to the Solid CD, hath a Proportion triplicate of that which the Side AE has to the Side CF.

For, produce AE, GE, HE, to EK, EL, EM; and make EK equal to CF, and EL to FN, and EM to FR; and let the Parallelogram KL, and likewiſe the Solid KO, be completed. Then, becauſe the two Sides KE, EL, are equal to the two Sides CF, FN, and the Angle KEL equal to the Angle CFN; ſince the Angle AEG is alſo equal to the Angle CFN, becauſe of the Similarity of the Solids AB, CD, the Parallelogram KL ſhall be ſimilar and equal to the Parallelogram CN. For the ſame Reaſon, the Parallelogram KM is equal and ſimilar to the Parallelogram CR, and the Parallelogram OE to DF. Therefore three Parallelograms of the Solid KO are equal and ſimilar to three Parallelograms of the Solid CD: But thoſe three Parallelograms are* equal and ſimilar to the three oppoſite Parallelograms. Therefore the whole Solid KO is equal and ſimilar to the whole Solid CD. Let the Parallelogram GK be completed, as alſo the Solids EX, LP, upon the Baſes GK, KL, having the ſame Altitude as AB: And ſince, becauſe of the Similarity of the Solids AB and CD, it is as AE is to CF, ſo is EG to FN; and ſo EH to FR; and FC is equal to EK, and FN to EL, and FR to EM; it ſhall be as AE is to EK,

* 24 *of this.*

E K, ſo is † the Parallelogram A G to the Parallelo- † 1. 6.
gram G K; but as G E is to E L, ſo is G K to K L; and as H E is † to E M, ſo is P E to K M. Therefore, as the Parallelogram A G is to the Parallelogram G K, ſo is G K to K L, and P E to K M. But as
A G is to G K, ſo is ‡ the Solid A B to the Solid E X; ‡ 32 *of this.*
and as G K is to K L, ſo is the Solid E X to the Solid P L; and as P E is to K M, ſo is the Solid P L to the Solid K O. Therefore, as the Solid A B is to the
Solid E X, ſo is * E X to P L, and P L to K O. But * 11. 5.
if four Magnitudes be continually proportional, the
firſt to the fourth hath † a triplicate Proportion of † *Def.* 11. 5.
that which it has to the ſecond. Therefore alſo the Solid A B, to the Solid K O, hath a triplicate Proportion of that which A B has to E X: But as A B is to E X, ſo is the Parallelogram A G to the Parallelogram G K; and ſo is the Right Line A E to the Right Line E K. Wherefore the Solid A B, to the Solid K O, hath a Proportion triplicate of that which A E has to E K; but the Solid K O is equal to the Solid C D, and the Right Line E K equal to the Right Line C F. Therefore the Solid A B, to the Solid C D, has a Proportion triplicate of that which the homologous Side A E has to the homologous Side C F; *which was to be demonſtrated.*

Coroll. From hence it is manifeſt, if four Right Lines be proportional, as the firſt is to the fourth, ſo is a ſolid Parallelopipedon deſcribed upon the firſt, to a ſimilar ſolid Parallelopidon, alike ſituate, deſcribed upon the ſecond; becauſe the firſt to the fourth, has a Proportion triplicate of that which it has to the ſecond.

PROPOSITION XXXIV.

THEOREM.

The Baſes and Altitudes of equal ſolid Parallelopipedons, are reciprocally proportional; and thoſe ſolid Parallelopepidons, whoſe Baſes and Altitudes are reciprocally proportional, are equal.

LET AB, CD, be equal ſolid Parallelopipedons, I ſay, their Baſes and Altitudes are reciprocally proportional; that is, as the Baſe EH is to the Baſe NP, ſo is the Altitude of the Solid CD to the Altitude of the Solid AB.

Firſt, let the inſiſtent Lines AG, EF, LB, HK, CM, NX, OD, PR, be at Right Angles to their Baſes. I ſay, as the Baſe EH is to the Baſe NP, ſo is CM to AG. For, if, the Baſe EH be equal to the Baſe NP, and the Solid AB is equal to the Solid CD, the Altitude CM ſhall alſo be equal to the Altitude AG: For, if when the Baſes EH, NP, are equal, the Altitudes AG, CM, are not ſo; then the Solid AB will not be equal to the Solid CD, but it is put equal to it. Therefore the Altitude CM is not unequal to the Altitude AG, and ſo they are neceſſarily equal to one another; and conſequently, as the Baſe EH is to the Baſe NP, ſo ſhall CM be to AG. But, now let the Baſe EH be unequal to the Baſe NP, and let EH be the greater: Then, ſince the Solid AB is equal to the Solid CD, CM is greater than AG; for otherwiſe it would follow, that the Solids AB, CD, are not equal, which are put ſuch. Therefore make CT equal to AG, and complete the ſolid Parallelopipedon VC upon the Baſe NP, having the Altitude CT. Then, becauſe the Solid AB is equal to the Solid CD, and VC is ſome other Solid;
* 7. 5. and ſince equal Magnitudes have * the ſame Proportion to the ſame Magnitude, it ſhall be as the Solid AB is to the Solid CV, ſo is the Solid CD to the Solid CV; but as the Solid AB is to the Solid CV,
† 32 *of this.* ſo is † the Baſe EH to the Baſe NP; for AB, CV, are Solids having equal Altitudes. And as the Solid
‡ 25 *of this.* CD is to the Solid CV, ſo is ‡ the Baſe MP to the Baſe

Baſe PT, and ſo is MC to CT. Therefore, as the Baſe EH is to the Baſe NP, ſo is MC to CT; but CT is equal to AG. Wherefore, as the Baſe EH is to the Baſe NP, ſo is MC to AG. Therefore the Baſes and Altitudes of the equal ſolid Parallelepipedons AB, CD, are reciprocally proportional.

Now, let the Baſes and Altitudes of the ſolid Parallelepipedons AB, CD, be reciprocally proportional; that is, let the Baſe EH be to the Baſe NP, as the Altitude of the Solid CD is to the Altitude of the Solid AB. I ſay, the Solid AB is equal to the Solid CD.

For, let again the inſiſtent Lines be at Right Angles, to the Baſes; then, if the Baſe EH be equal to the Baſe NP, and EH is to NP, as the Altitude of the Solid CD is to the Altitude of the Solid AB; the Altitude of the Solid CD ſhall be equal to the Altitude of the Solid AB. But ſolid Parallelepipedons, that ſtand upon equal Baſes, and have the ſame Altitude, are* equal to each other. Therefore the Solid AB is equal to the Solid CD. *31 *of this.*

But now let the Baſe EH not be equal to the Baſe NP, and let EH be the greater; then the Altitude of the Solid CD is greater than the Altitude of the Solid AB; that is, CM is greater than AG. Again, put CT equal to AG, and complete the Solid CV as before: And then, becauſe the Baſe EH is to the Baſe NP, as MC is to AG, and AG is equal to CT; it ſhall be as the Baſe EH is to the Baſe NP, ſo is MC to CT; but as the Baſe EH is to the Baſe NP, ſo is the Solid AB to the Solid VC; for the Solids AB, CV, have equal Altitudes: And as MC is to CT, ſo is the Baſe MP to the Baſe PT, and ſo the Solid CD to the Solid CV. Therefore, as the Solid AB is to the Solid CV, ſo is the Solid CD to the Solid CV: But ſince each of the Solids AB, CD, has the ſame Proportion to CV, the Solid AB ſhall be equal to the Solid CD; *which was to be demonſtrated.*

Now let the inſiſtent Lines FE, BL, GA, KH, XN, DO, MC, RP, not be at Right Angles to the Baſes; and from the Points F, G, B, K, X, M, D, R, let there be drawn Perpendiculars to the Planes of the Baſes EH, NP, meeting the ſame in the

Points S, T, Y, V, Q, Z, Ω, Φ, and complete the Solids FV, XΩ. Then, I ſay, if the Solids AB, CD, be equal, their Baſes and Altitudes are reciprocally proportional; *viz.* as the Baſe EH is to the Baſe NP, ſo is the Altitude of the Solid CD to the Altitude of the Solid AB.

For, becauſe the Solid AB is equal to the Solid CD, and the Solid AB is * equal to the Solid BT; for they ſtand upon the ſame Baſe, have the ſame Altitude, and their inſiſtent Lines are not in the ſame Right Lines, and the Solid DC is * equal to the Solid DZ, ſince they ſtand upon the ſame Baſe XR, have the ſame Altitude, and their inſiſtent Lines are not in the ſame Right Lines; the Solid BT ſhall be equal to the Solid DZ; but the Baſes and Altitudes of thoſe equal Solids, whoſe Altitudes are at Right Angles to their Baſes, are † reciprocally proportional. Therefore, as the Baſe FK is to the Baſe XR, ſo is the Altitude of the Solid DZ, to the Altitude of the Solid BT; but the Baſe FK is equal to the Baſe EH, and the Baſe XR to the Baſe NP. Wherefore, as the Baſe EH is to the Baſe NP, ſo is the Altitude of the Solid DZ to the Altitude of the Solid BT; but the Solids DZ, DC, have the ſame Altitude, and ſo have the Solids BT, BA. Therefore the Baſe EH is to the Baſe NP, as the Altitude of the Solid DC is to the Altitude of the Solid AB; and ſo the Baſes and Altitudes of equal Solids are reciprocally proportional.

* 30 *of this.*

† *From what has been before proved.*

Again, let the Baſes and Altitudes of the ſolid Parallelepipedons AB, CD, be reciprocally proportional; *viz.* as the Baſe EH is to the Baſe NP, ſo let the Altitude of the Solid CD be to the Altitude of the Solid AB. I ſay, the Solid AB is equal to the Solid CD.

For the ſame Conſtruction remaining, becauſe the Baſe EH is to the Baſe NP, as the Altitude of the Solid CD is to the Altitude of the Solid AB; and ſince the Baſe EH is equal to the Baſe FK, and NP to XR, it ſhall be as the Baſe FK is to the Baſe XR, ſo is the Altitude of the Solid CD to the Altitude of the Solid AB; but the Altitudes of the Solids AB, BT, are the ſame; as alſo of the Solids CD, DZ. Therefore the Baſe FK is to the Baſe XR, as the

Altitude of the Solid DZ is to the Altitude of the Solid BT; wherefore the Baſes and Altitudes of the ſolid Parallelepipedons BT, DZ, are reciprocally proportional; but thoſe ſolid Parallelepipedons, whoſe Altitudes are at Right Angles to their Baſes, and the Baſes and Altitudes are reciprocally proportional, are equal to each other. Bnt the Solid BT is equal to the Solid BA; for they ſtand upon the ſame Baſe FK, have the ſame Altitude, and their inſiſtent Lines are not in the ſame Right Lines; and the Solid DZ is alſo equal to the Solid DC, ſince they ſtand upon the ſame Baſe XR, have the ſame Altitude, and their inſiſtent Lines are not in the ſame Right Lines. Therefore the Solid AB is equal to the Solid CD; *which was to be demonſtrated.*

PROPOSITION XXXV.

THEOREM.

If there be two plain Angles equal, and from the Vertices of thoſe Angles two Right Lines be elevated above the Planes, in which the Angles are, containing equal Angles with the Lines firſt given, each to its correſpondent one; and if in thoſe elevated Lines any Points be taken, from which Lines be drawn perpendicular to the Planes in which the Angles firſt given are, and Right Lines be drawn to the Angles firſt given from the Points made by the Perpendiculars in the Planes, thoſe Right Lines will contain equal Angles with the elevated Lines.

LET BAC, EDF, be two equal Right-lined Angles; and from A, D, the Vertices of thoſe Angles, let two Right Lines, AG, DM, be elevated above the Planes of the ſaid Angles, making equal Angles with the Lines firſt given, each to its correſpondent one; *viz.* the Angle MDE equal to the Angle GAB, and the Angle MDF to the Angle GAC; and take any Points G and M in the Right Lines AG, DM, from which let GL, MN, be drawn perpendicular to the Planes paſſing thro' BAC, EDF, meeting the ſame in the Points L, N; and

join LA, ND. I ſay, the Angle GAL is equal to
the Angle MDN.

Make AH equal to DM, and thro' H let HK be
drawn parallel to GL; but GL is perpendicular to
the Plane paſſing thro' BAC. Therefore HK ſhall
*8 *of this.* be * alſo perpendicular to the Plane paſſing thro' BAC.
Draw from the Points K, N, to the Right Lines AB,
AC, DE, DF, the Perpendiculars KB, KC, NE,
NF; and join HC, CB, MF, FE. Then, becauſe
† 47. 1. the Square of HA is † equal to the Squares of HK,
KA, and the Squares of KC, CA, are † equal to
the Square of KA; the Square of HA ſhall be equal
to the Squares of HK, KC, CA; but the Square of
HC is equal to the Squares of HK, KC. Therefore
the Square of HA will be equal to the Squares of
‡ 48. 1. HC and CA; and ſo the Angle HCA is ‡ a Right
Angle. For the ſame Reaſon, the Angle DFM is
alſo a Right Angle. Therefore the Angle ACH is
equal to DFM; but the Angle HAC is alſo equal
to the Angle MDF. Therefore the two Triangles
MDF, HAC, have two Angles of the one equal to
two Angles of the other, each to each, and one Side
of the one equal to one Side of the other, *viz.* that
which is ſubtended by one of the equal Angles, that
is, the Side HA, equal to DM; and ſo the other
* 26. 1. Sides of the one ſhall be * equal to the other Sides
of the other, each to each. Wherefore AC is equal
to DF. In like manner we demonſtrate, that AB
is equal to DE; for let HB, ME, be joined. Then,
becauſe the Square of AH is equal to the Squares of
AK and KH; and the Squares of AB, BK, are
equal to the Square of AK; the Squares of AB, BK,
KH, will be equal to the Square of AH, but the
Square of BH is equal to the Squares of BK, KH;
for the Angle HKB is a Right Angle, becauſe HK
is perpendicular to the Plane paſſing through BAC.
Therefore the Square of AH is equal to the Squares
† 48. 1. of AB, BH. Wherefore the Angle ABH is † a Right
Angle. For the ſame Reaſon, the Angle DEM
is alſo a Right Angle. And the Angle BAH is
equal to the Angle EDM, for ſo it is put; and AH
is equal to DM. Therefore AB is † alſo equal to
DE. And ſo, ſince AC is equal to DF, and AB
to DE, the two Sides CA, AB, ſhall be equal to
the

the two Sides FD, DE; but the Angle BAC is equal to the Angle FDE. Therefore the Baſe BC is* equal to the Baſe EF, the Triangle to the Triangle, and the other Angles to the other Angles. Wherefore the Angle ACB is equal to the Angle DFE; but the Right Angle ACK is equal to the Right Angle DFN; and therefore the remaining Angle BCK is equal to the remaining Angle EFN. For the ſame Reaſon, the Angle CBK is equal to the Angle FEN; and ſo, becauſe BCK, EFN, are two Triangles, having two Angles equal to two Angles each to each, and one Side equal to one Side, which is at the equal Angles, *viz.* BC equal to EF; therefore they ſhall have the other Sides equal to the other Sides. Therefore CK is equal to FN; but AC is equal to DF. Therefore the two Sides AC, CK, are equal to the two Sides DF, FN, and they contain Right Angles; conſequently the Baſe AK is equal to the Baſe DN. And ſince AH is equal to DM, the Square of AH ſhall be equal to the Square of DM; but the Squares of AK, KH, are equal to the Square of AH; for the Angle AKH is a Right Angle, and the Squares DN, NM, are equal to the Square of DM, ſince the Angle DNM is a Right Angle. Therefore the Squares of AK, KH, are equal to the Squares of DN, NM; of which the Square of AK is equal to the Square of DN. Wherefore the Square of KH remaining, is equal to the remaining Square of NM; and ſo the Right Line HK is equal to MN. And ſince the two Sides HA, AK, are equal to the two Sides MD, DN, each to each, and the Baſe HK has been proved equal to the Baſe NM, the Angle HAK ſhall be† equal to the Angle MDN; *which was to be demonſtrated.*

* 4. 1.

† 8. 1.

Coroll. From hence it is manifeſt, that if there be two Right-lined plain Angles equal, from whoſe Points equal Right Lines be elevated on the Planes of the Angles, containing equal Angles with the Lines firſt given, each to each; Perpendiculars drawn from the extreme Points of thoſe elevated Lines to the Planes of the Angles firſt given, are equal to one another.

PROPOSITION XXXVI.

THEOREM.

If three Right Lines be proportional, the ſolid Parallelepipedon made of them, is equal to the ſolid Parallelepipedon made of the middle Line, if it be an equilateral one, and equiangular to the aforeſaid Parallelepipedon.

LET three Right Lines A, B, C, be proportional; *viz.* Let A be to B, as B is to C. I ſay, the Solid made of A, B, C, is equal to the equilateral Solid made of B, equiangular to that made on A, B, C.

Let E be a ſolid Angle contained under the three plain Angles D E G, G E F, F E D; and make D E, G E, E F, each equal to B, and complete the ſolid Parallelepipedon E K. Again, put L M equal to A, and at the Point L, at the Right Line L M, make* a ſolid Angle contained under the plain Angles N L X, X L M, M L N, equal to the ſolid Angle E; and make L N equal to B, and L X to C. Then, becauſe A is to B, as B is to C, and A is equal to L M, and B to L N, E F, E G, or E D, and C to L X; it ſhall be as L M is to E F, ſo is G E to L X. And ſo the Sides about the equal Angles M L X, G E F, are reciprocally proportional. Wherefore the Parallelogram M X is† equal to the Parallelogram G F. And ſince the two plain Angles G E F, X L M, are equal, and the Right Lines L N, E D, being equal, are erected at the angular Points containing equal Angles with the Lines firſt given, each to each; the Perpendiculars drawn‡ from the Points N D, to the Planes drawn thro' X L M, G E F, are equal one to another. Therefore the Solids L H, E K, have the ſame Altitude; but ſolid Parallelepipedons that have equal Baſes, and the ſame Altitude, are* equal to each other. Therefore the Solid H L is equal to the Solid E K. But the Solid H L is that made of three Right Angles A, B, C, and the Solid E K that made of the Right Line B. Therefore, *if three Right Lines be proportional, the ſolid Parallelepipedon made of them, is equal to the ſolid Parallelepipedon made of the middle Line, if it be an* equi-

* 26 *of this.*

† 14. 6.

‡ *Cor.* 35. *of this.*

* 31 *of this.*

equilateral one, and equiangular to the aforeſaid Parallelepipedon; which was to be demonſtrated.

PROPOSITION XXXVII.

THEOREM.

If four Right Lines be proportional, the ſolid Parallelepipedons ſimilar, and in like manner deſcribed from them, ſhall be proportional. And if the ſolid Parallelepipedons, being ſimilar, and alike deſcribed, be proportional, then the Right Lines they are deſcribed from, ſhall be proportional.

LET the four Right Lines AB, CD, EF, GH, be proportional; *viz.* let AB be to CD, as EF is to GH; and let the ſimilar and alike ſituate Parallelepipedons KA, LC, ME, NG, be deſcribed from them. I ſay, KA is to LC, as ME is to NG.

For, becauſe the ſolid Parallelepipedon KA is ſimilar to LC, therefore KA to LC ſhall have *a Proportion triplicate of that which AB has to CD. For the ſame Reaſon, the Solid ME to NG will have a triplicate Proportion of that which EF has to GH. But AB is to CD, as EF is to GH. Therefore AK is to LC, as ME is to NG. And if the Solid AK be to the Solid LC, as the Solid ME is to the Solid NG, I ſay, as the Right Line AB is to the Right Line CD, ſo is the Right EF to the Right Line GH. For, becauſe AK to LC has †a Proportion triplicate of that which AB has to CD, and ME to NG has a Proportion triplicate of that which EF has to GH, and ſince AK is to LC, as ME is to NG, it ſhall be as AB is to CD, ſo is EF to GH. Therefore, *if four Right Lines be proportional, the ſolid Parallelepipedons ſimilar, and in like manner deſcribed from them, ſhall be proportional. And if the ſolid Parallelepipedons, being ſimilar and alike deſcribed, be proportional, then the Right Lines they are deſcribed from ſhall be proportional*; which was to be demonſtrated.

* 33 *of this.*

† 33 *of this.*

PROPOSITION XXXVIII.

THEOREM.

If a Plane be perpendicular to a Plane, and a Line be drawn from a Point in one of the Planes perpendicular to the other Plane, that Perpendicular ſhall fall in the common Section of the Planes.

LET the Plane CD be perpendicular to the Plane AB, let their common Section be AD, and let ſome Point E be taken in the Plane CD. I ſay, a Perpendicular, drawn from the Point E, to the Plane AB, falls on AD.

For if it does not, let it fall without the ſame, as EF meeting the Plane AB in the Point F; and from the Point F, let FG be drawn in the Plane AB perpendicular to AD; this ſhall be * perpendicular to the Plane CD; and join EG. Then, becauſe FG is perpendicular to the Plane CD, and the Right Line EG in the Plane of CD touches it: The Angle FGE ſhall be † a Right Angle. But EF is alſo at Right Angles to the Plane AB; therefore the Angle EFG is a Right Angle. And ſo two Angles of the Triangle EFG are equal to two Right Angles; which is ‡ abſurd. Wherefore a Right Line, drawn from the Point E, perpendicular to the Plane AB, does not fall without the Right Line AD: And ſo it muſt neceſſarily fall on it. Therefore, *if a Plane be perpendicular to a Plane, and a Line be drawn from a Point in one of the Planes perpendicular to the other Plane, that Perpendicular ſhall fall in the common Section of the Planes*; which was to be demonſtrated.

* *Def.* 4. *of this.*

† *Def.* 3. *of this.*

‡ 17. 1.

PRO-

PROPOSITION XXXIX.

THEOREM.

If the Sides of the oppoſite Planes of a ſolid Parallelepipedon be divided into two equal Parts, and Planes be drawn thro' their Sections; the common Section of the Planes, and the Diameter of the ſolid Parallelepipedon, ſhall divide each other into two equal Parts.

LET the Sides of CF, AH, the oppoſite Planes of the ſolid Parallelepipedon AF, be cut in half in the Points K, L, M, N, X, O, P, R; and let the Planes KN, XR, be drawn thro' the Sections: Alſo let YS be the common Section of the Planes, and DG the Diameter of the ſolid Parallelepipedon. I ſay, YS, DG, biſect each other; that is, YT is equal to TS, and DT to TG.

For, join DY, YE, BS, SG. Then, becauſe DX is parallel to OE, the alternate Angles DXY, YOE, are * equal to one another. And becauſe DX is equal * 29. 1. to OE, and YX to YO, and they contain equal Angles, the Baſe DY ſhall be † equal to the Baſe YE; † 4. 1. and the Triangle DXY to the Triangle YOE, and the other Angles equal to the other Angles: Therefore the Angle XYD is equal to the Angle OYE; and ſo DYE is ‡ a Right Line. For the ſame Reaſon ‡ 14. 1. BSG is alſo a Right Line, and BS is equal to SG. Then, becauſe CA is equal and parallel to DB, as alſo to EG, DB ſhall be equal and parallel to EG; and the Right Lines DE, GB, join them: Therefore DE is * parallel to BG, and D, Y, G, S, are Points * 33. 1. taken in each of them; and DG, YS, are joined. Therefore DG, YS, are † in one Plane. And ſince DE † 7 of this. is parallel to BG, the Angle EDT ſhall be * equal to * 29. 1. the Angle BGT; for they are alternate. But the Angle DTY is ‡ equal to the Angle GTS. Therefore ‡ 15. 1. DTY, GTS, are two Triangles having two Angles of the one equal to two Angles of the other, as likewiſe one Side of the one equal to one Side of the other, *viz.* the Side DY equal to the Side GS: For they are Halves of DE, BG: Therefore they ſhall have the other Sides of one equal to the other Sides of the other; and ſo DT is equal to TG, and YT to TS. Wherefore, *if the Sides of the oppoſite Planes of a ſolid Parallelepipedon be divided into two equal Parts, and Planes be drawn thro' their Sections; the common Section of the Planes, and the Diameter of the ſolid Parallelepipedon, ſhall divide each other into two equal Parts*; which was to be demonſtrated.

PROPOSITION XL.

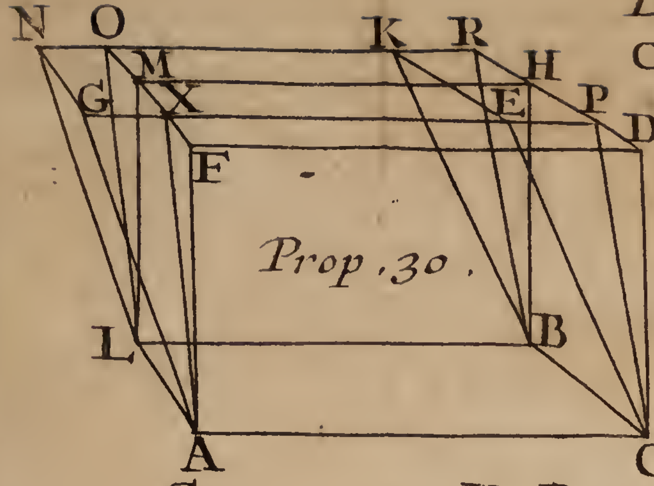
THEOREM.

If of two triangular Priſms, one ſtanding on a Baſe, which is a Parallelogram, and the other on a Triangle, if their Altitudes from theſe Baſes are equal, and the Parallelogram double to the Triangle; then thoſe Priſms are equal to each other.

LET ABCDEF, GHKLMN, be two Priſms of equal Altitude, the Baſe of one of which is the Parallelogram AF, and that of the other, the Triangle GHK; and let the Parallelogram AF be double to the Triangle GHK. I ſay, the Priſm ABCDEF is equal to the Priſm GHKLMN.

For, complete the Solids AX, GO. Then, becauſe the Parallelogram AF is double to the Triangle
* 41. 1. GHK, and ſince the Parallelogram HK is * double to the Triangle GHK, the Parallelogram AF ſhall be equal to the Parallelogram HK. But ſolid Parallelopipedons, that ſtand upon equal Baſes, and have the ſame
† 31 *of this.* Altitude, are † equal to one another. Therefore the Solid AX is equal to the Solid GO. But the Priſm ABCDEF is half the Solid AX, and the Priſm
‡ 28 *of this.* GHKLMN is ‡ half the Solid GO. Therefore the Priſm ABCDEF is equal to the Priſm GHKLMN. Wherefore, *if there be two Priſms having equal Altitudes, the Baſe of one of which is a Parallelogram, and that of the other a Triangle, and if the Parallelogram be double to the Triangle, the ſaid Priſms ſhall be equal to each other.*

The END *of the* ELEVENTH BOOK.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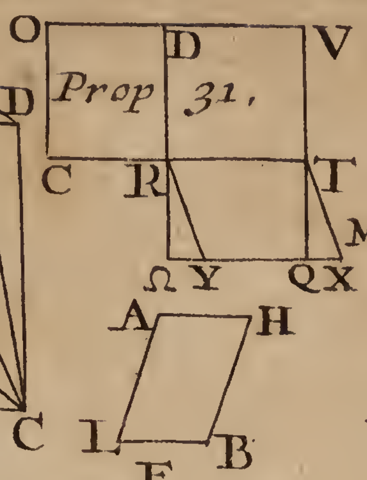
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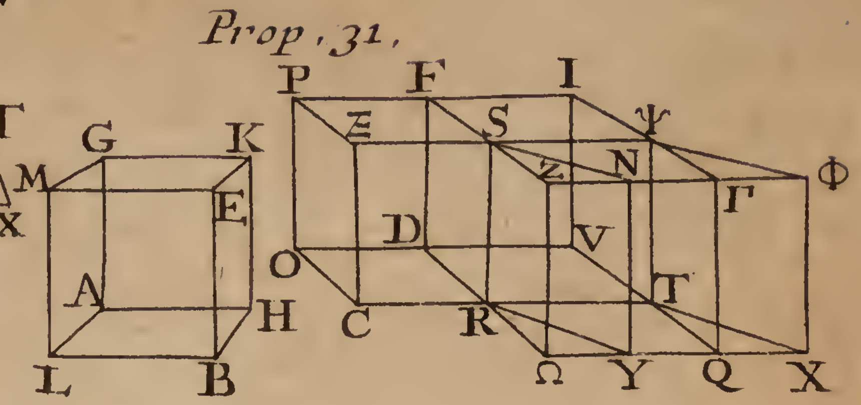
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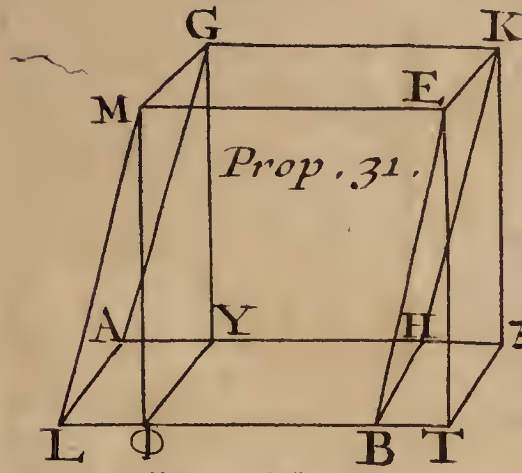
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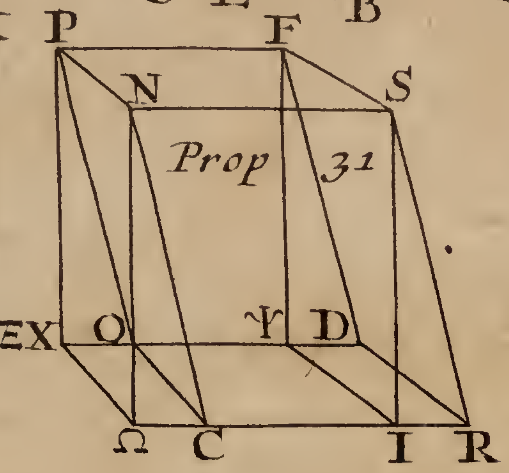
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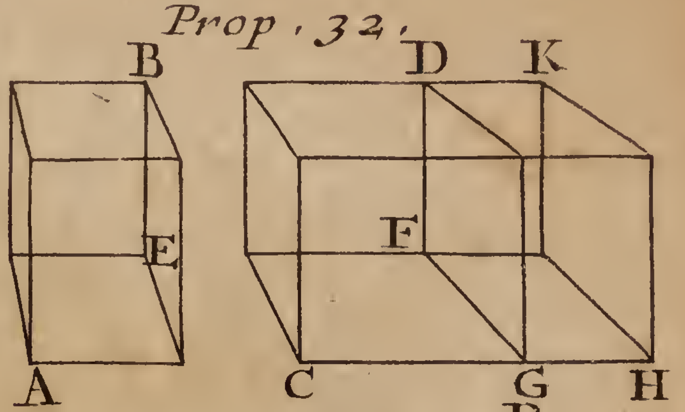
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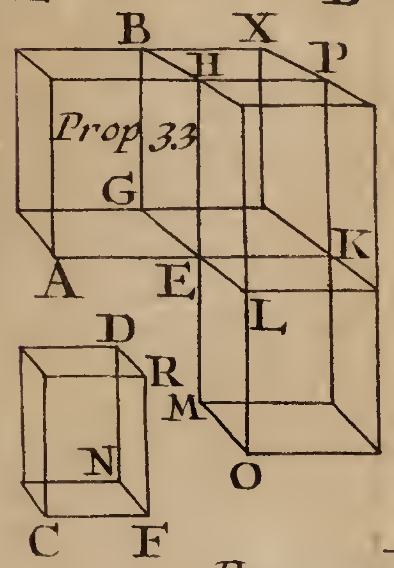
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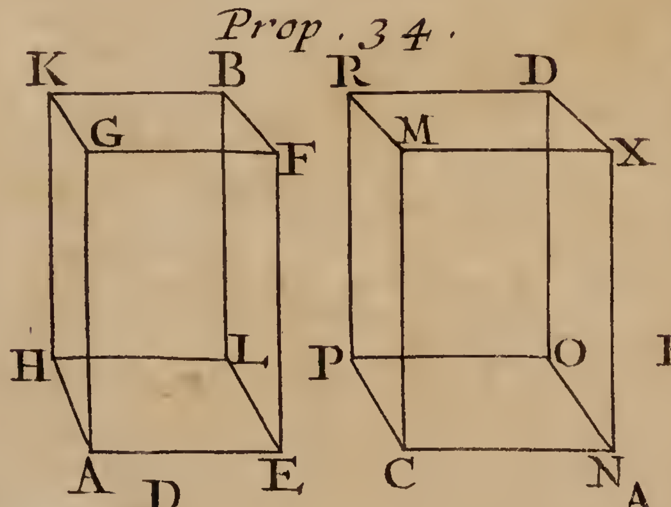
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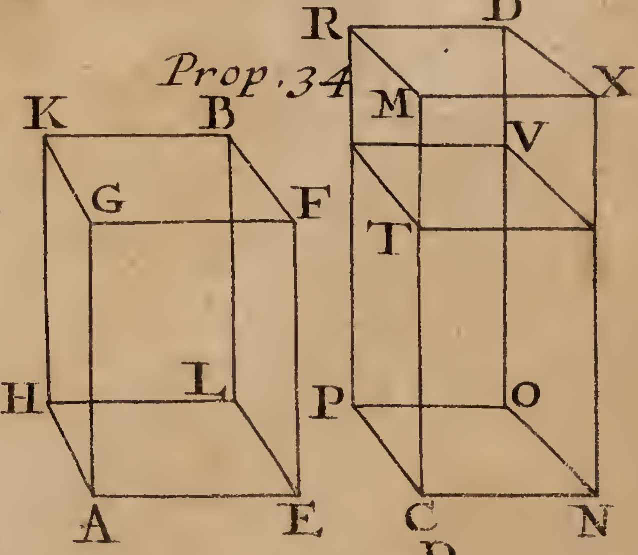
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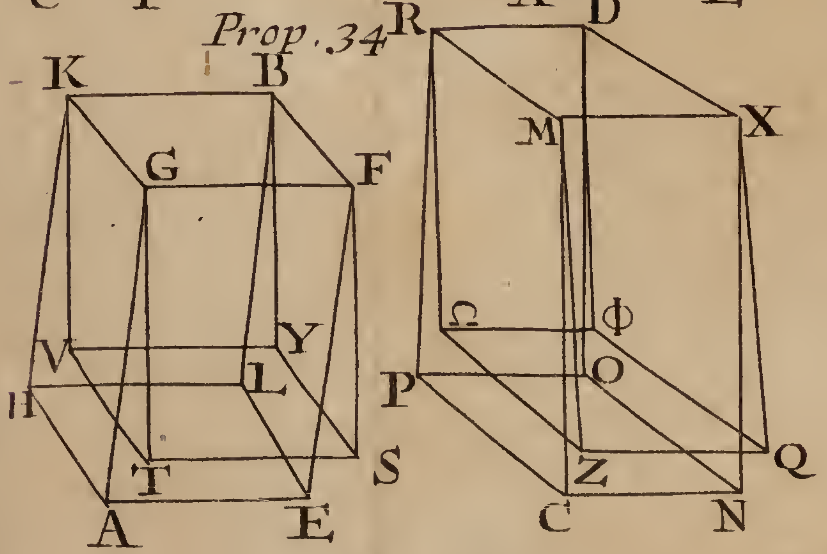
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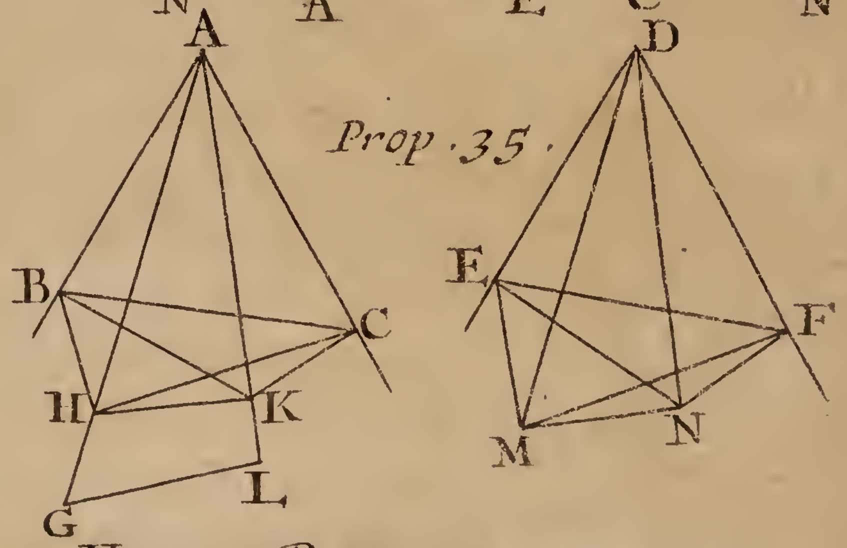
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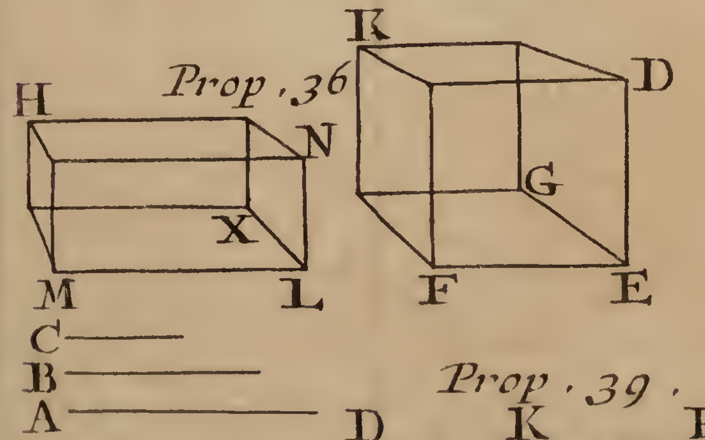
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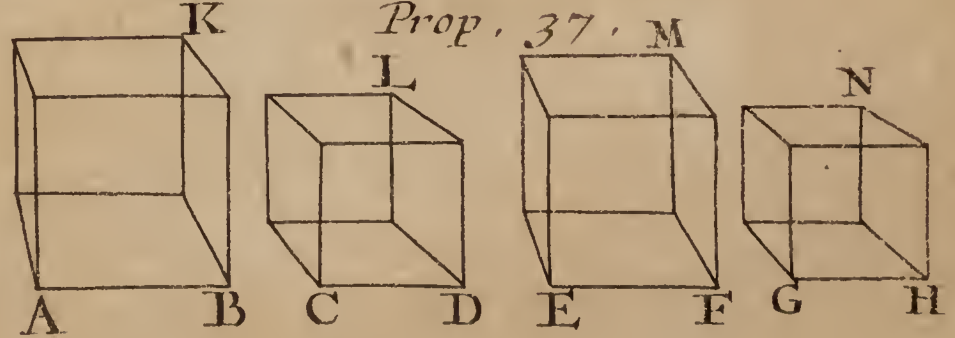
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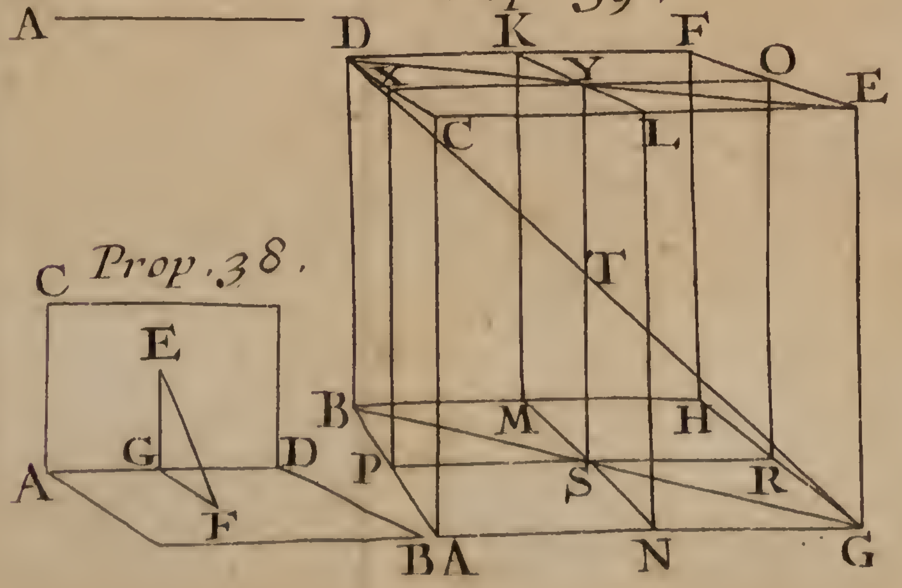
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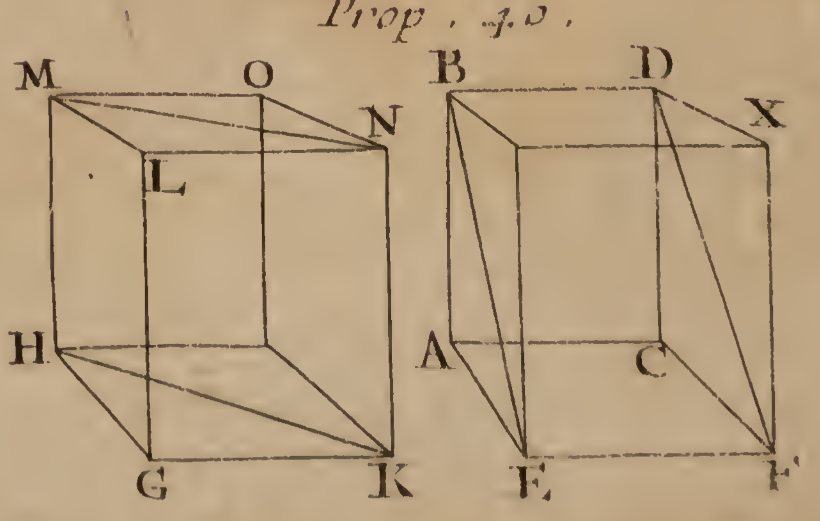
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EUCLID's

ELEMENTS.

BOOK XII.

PROPOSITION I.

THEOREM.

Similar Polygons, inſcribed in Circles, are to one another as the Squares of the Diameters of the Circles.

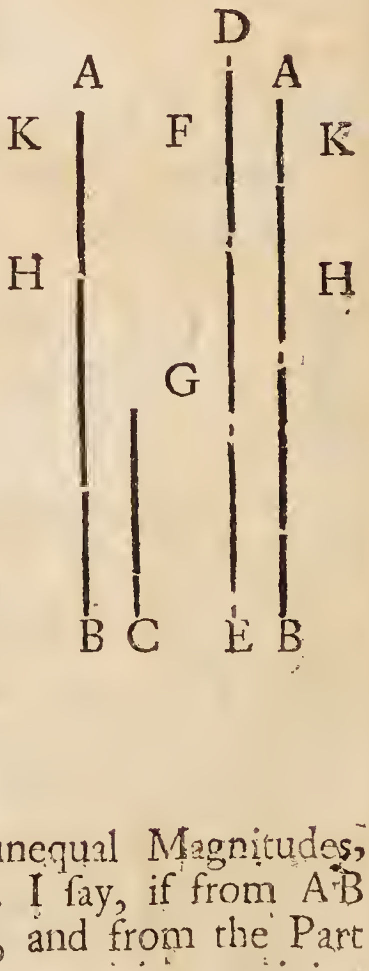
LET ABCDE, FGHKL, be Circles, wherein are inſcribed the ſimilar Polygons ABCDE, FGHKL; and let BM, GN, be Diameters of the Circles. I ſay, as the Square of BM is to the Square of GN, ſo is the Polygon ABCDE to the Polygon FGHKL.

For join BE, AM, GL, FN. Then, becauſe the Polygon ABCDE is ſimilar to the Polygon FGHKL, the Angle BAE is equal to the Angle GFL; and BA is to AE, as GF is to FL. Therefore the two Triangles BAE, GFL, have one Angle of the one equal to one Angle of the other, *viz.* the Angle BAE equal to the Angle GFL, and the Sides about the equal Angles proportional. Wherefore the Triangle ABE is * equiangular to the Triangle FGL; * 6. 6.
and

and ſo the Angle AEB is equal to the Angle FLG;
† 21. 3. But the Angle AEB is † equal to the Angle AMB;
for they ſtand on the ſame Circumference; and the
Angle FLG is † equal to the Angle FNG. There-
fore the Angle AMB is equal to the Angle FNG.
‡ 31. 3. But the Right Angle BAM is ‡ equal to the Right
Angle GFN. Wherefore the other Angle ſhall be
equal to the other Angle. And ſo the Triangle AMB
is equiangular to the Triangle FGN; and conſe-
* 4. 6. quently * as BM is to GN, ſo is BA to GF. But
the Proportion of the Square of BM to the Square
of GN, is duplicate of the Proportion of BM to
GN; and the Proportion of the Polygon ABCDE
† 20. 6. to the Polygon FGHKL, is † duplicate of the Pro-
portion of BA to GF. Wherefore, as the Square of
BM is to the Square of GN, ſo is the Polygon
ABCDE to the Polygon FGHKL. Therefore,
*ſimilar Polygons, inſcribed in Circles, are to one an-
other as the Squares of the Diameters of the Circles*;
which was to be demonſtrated.

LEMMA.

If there be two unequal Magnitudes propos'd, and from the greater be taken a Part greater than its Half; and if from what remains there be again taken a Part greater than half this Remainder; and again from this laſt Remainder a Part greater than its Half; and if this be done continually, there will remain at laſt a Magnitude that ſhall be leſs than the leſſer of the propos'd Magnitudes.



LET AB and C be two unequal Magnitudes, whereof AB is the greater. I ſay, if from AB be taken a greater Part than half, and from the Part remaining there be again taken a Part greater than its Half, and this be done continually, there will remain a Magnitude at laſt, that ſhall be leſs than the Magnitude C.

For C being ſome Number of Times multiplied, will become greater than the Magnitude AB. Let it be multiplied, and let DE be a Multiple of C greater than AB. Divide DE into Parts DF, FG, GE, each equal to C; and take BH, a Part greater than half of AB, from AB, and again from AH the Part HK greater than half AH, and from AK a Part greater than half AK, and ſo on, until the Diviſions, that are in AB are equal in Number to the Diviſions in DE. Therefore let the Diviſions AK, KH, HB, be equal in Number to the Diviſions DF, FG, GE. Then, becauſe DE is greater than AB, and the Part EG is taken from ED, being leſs than half thereof, and the Part BH greater than half of AB is taken from it, the Part remaining GD ſhall be greater than the Part remaining HA. Again, becauſe GD is greater than HA; and GF, being half of GD, is taken from the ſame; and HK, being greater than half HA, is taken from this likewiſe; the Part remaining FD ſhall be greater than the Part remaining AK; but FD is equal to C. Therefore C is greater than AK; and ſo the Magnitude AK is leſſer than C. Therefore the Magnitude AK, being the Part remaining of the Magnitude AB, is leſs than the leſſer propos'd Magnitude C; *which was to be demonſtrated.* If the Halves of the Magnitudes ſhould have been taken, we demonſtrate this after the ſame manner. *This is the firſt Propoſition of the tenth Book.*

PROPOSITION II.

THEOREM.

Circles are to each other as the Squares of their Diameters.

LET ABCD, EFGH, be Circles, whoſe Diameters are BD, FH. I ſay, as the Square of BD is to the Square of FH, ſo the Circle ABCD to the Circle EFGH.

For if it be not ſo, the Square of B D ſhall be to the Square of F H, as the Circle A B C D is to ſome Space either leſs or greater than the Circle E F G H. Firſt, let it be to a Space S, leſs than the Circle E F G H, and let the Square E F G H be deſcribed therein.

This Square E F G H will be greater than half the Circle E F G H; becauſe, if we drawn Tangents to the Circle thro' the Points E, F, G, H, the Square E F G H will be half that deſcribed about the Circle; but the Circle is leſs than the Square deſcribed about it. Therefore the Square E F G H is greater than half the Circle E F G H. Let the Circumferences E F, F G, G H, H E, be biſected in the Points K, L, M, N; and join E K, K F, F L, L G, G M, M H, H N, N E. Then each of the Triangles E K F, F L G, G M H, H N E, will be * greater than one half of the Segment of the Circle it ſtands in: Becauſe, if Tangents at the Circle be drawn thro' the Points K, L, M, N, and the Parallelograms that are on the Right Lines E F, F G, G H, H E, be completed, each of the Triangles E K F, F L G, G M H, H N E, is half of each of the correſponding Parallelograms; but the Segment is leſs than the Parallelogram. Wherefore each of the Triangles E K F, F L G, G M H, H N E, is greater than one half of the Segment of the Circle in which it ſtands. Therefore, if theſe Circumferences be again biſected, and Right Lines be drawn joining the Points of Biſection, and you do thus continually, there will at laſt remain Segments of the Circle, that ſhall be leſs than the Exceſs, by which the Circle E F G H exceeds the Space S. For it is demonſtrated in the foregoing *Lemma*, that if there be two unequal Magnitudes propoſed, and if from the greater a Part greater than half be taken from it, and again from the Part remaining a Part greater than half be taken, and you do this continually; there will at laſt remain a Magnitude that will be leſs than the leſſer propoſed Magnitude. Let the Segments of the Circle E F G H on the Right Lines E K, K F, F L, L G, G M, M H, H N, N E, be thoſe which are leſs than the Exceſs, whereby the Circle E F G H exceeds the Space S; and then the remaining Polygon E K F L G M H N ſhall be greater than the Space S. Alſo deſcribe the Polygon A X B O C P D R in the Circle A B C D, ſimilar to the Polygon

* 41. 1.

Polygon EKFLGMHN. Wherefore, as the Square
of BD is to the Square of FH, ſo is the Polygon AX
BOCPDR to * the Polygon EKFLGMHN. But * 1 *of this.*
as the Square of BD is to the Square of FH, ſo is the
Circle ABCD to the Space S. Wherefore, as the
Circle ABCD is to the Space S, ſo is † the Polygon † 11. 5.
AXBOCPDR to the Polygon EKFDGMHN.
But the Circle ABCD is greater than the Polygon in
it. Wherefore the Space S ſhall be † alſo greater
than the Polygon EKFLGMHN: But it is leſs ‡ ‡ *From the*
likewiſe; which is abſurd. Therefore the Square of *Hyp.*
BD to the Square of FH, is not as the Circle ABCD
to ſome Space leſs than the Circle EFGH. After
the ſame manner we likewiſe demonſtrate, that the
Square of FH to the Square of BD is not as the Circle
EFGH, to ſome Space leſs than the Circle
ABCD. Laſtly, I ſay, the Square of BD to the Square
of FH is not as the Circle ABCD, to ſome Space
greater than the Circle EFGH; for if it be poſſible,
let it be ſo, and let the Space S be greater than the
Circle EFGH; then ſhall it be (by Inverſions) as the
Square of FH is to the Square of BD, ſo is the Space
S to the Circle ABCD. But becauſe S is greater than
the Circle EFGH, the Space ſhall be to the Circle
ABCD, as the Circle EFGH is to ſome Space leſs
than the Circle ABCD. Therefore, as the Square
of FH is to the Square of BD, ſo is * the Circle * 11. 5.
EFGH to ſome Space leſs than the Circle ABCD;
which has been demonſtrated to be impoſſible. Wherefore
the Square of BD to the Square of FH, is not
as the Circle ABCD to ſome Space greater than the
Circle EFGH. But this alſo has been proved, that
the Square of BD to the Square of FH, is not as the
Circle ABCD to ſome Space leſs than the Circle
EFGH. Wherefore, as the Square of BD is to the
Square of FH, ſo ſhall the Circle ABCD be to the
Circle EFGH. Wherefore Circles are to each other
as the Squares of their Diameters; *which was to be
demonſtrated.*

PROPOSITION III.

THEOREM.

Every Pyramid having a triangular Baſe may be divided into two Pyramids, equal and ſimilar to one another, having triangular Baſes, and ſimilar to the whole Pyramid, and into two equal Priſms, which two Priſms are greater than the Half of the whole Pyramid.

LET there be a Pyramid, whoſe Baſe is the Triangle A B C; and Vertex the Point D. I ſay, the Pyramid A B C D may be divided into two Pyramids equal and ſimilar to one another, having triangular Baſes, and ſimilar to the Whole; and into two equal Priſms, which two Priſms are greater than the Half of the whole Pyramid.

For, biſect A B, B C, C A, A D, D B, D C, in the Points E, F, G, H, K, L; and join E H, E G, G H, H K, K L, L H, E K, K F, F G. Then, becauſe A E is equal to E B, and A H to H D, E H ſhall be * parallel to D B. For the ſame Reaſon, H K alſo is parallel to A B. Therefore H E B K is a Parallelogram; and ſo H K is † equal to E B; but E B is equal to A E. Therefore A E ſhall be alſo equal to H K; but A H is equal to H D. Wherefore the two Sides A E, A H, are equal to the two Sides K H, H D, each to each, and the Angle E A H is ‡ equal to the Angle K H D: Wherefore the Baſe E H is * equal to the Baſe K D: And ſo the Triangle A E H is equal and ſimilar to the Triangle H K D. For the ſame Reaſon, the Triangle A H G ſhall alſo be equal and ſimilar to the Triangle H L D. And becauſe the two Right Lines E H, H G, touching each other, are parallel to the two Right Lines K D, D L, touching each other, and not in the ſame Plane with them, they ſhall contain † equal Angles. Therefore the Angle E H G is equal to the Angle K D L. Again, becauſe the two Sides E H, H G, are equal to the two Sides K D, D L, each to each, and the Angle E H G is equal to the Angle K D L, the Baſe E G ſhall be * equal to the Baſe K L: And therefore the Triangle E H G is equal and ſimilar to the Triangle K D L. For the ſame Reaſon, the Triangle

* 2. 6.
† 34. 1.
‡ 29. 1.
* 4. 1.
† 10. 11.
* 4. 1.

angle A E G is alſo equal and ſimilar to the Triangle H K L. Wherefore the Pyramid whoſe Baſe is the Triangle A E G, and Vertex the Point H, is equal and ſimilar to the Pyramid whoſe Baſe is the Triangle H K L, and Vertex the Point D. And becauſe H K is drawn parallel to the Side A B of the Triangle A D B, the Triangle A D B ſhall be equiangular to the Triangle D K H, and they have their Sides proportional. Therefore the Triangle A D B is ſimilar to the Triangle D H K. And for the ſame Reaſon, the Triangle D B C is ſimilar to the Triangle D K L; and the Triangle A H G to the Triangle D H L. And ſince the two Right Lines B A, A C, touching each other, are parallel to the two Lines K H, H L, touching each other, not being in the ſame Plane with them, theſe ſhall contain equal Angles. Therefore the Angle B A C is equal to the Angle K H L. And B A is to A C, as K H is to H L. Wherefore the Triangle A B C is ſimilar to the Triangle H K L; and ſo the Pyramid, whoſe Baſe is the Triangle A B C, and Vertex the Point D, is ſimilar to the Pyramid, whoſe Baſe is the Triangle H K L, and Vertex the Point D. But the Pyramid, whoſe Baſe is the Triangle H K L, and Vertex the Point D, has been proved ſimilar to the Pyramid whoſe Baſe is the Triangle A E G, and Vertex the Point H. Therefore the Pyramid whoſe Baſe is the Triangle A B C, and Vertex the Point D, is ſimilar to the Pyramid whoſe Baſe is the Triangle A E G, and Vertex the Point H. Wherefore both the Pyramids A E G H, H K L D, are ſimilar to the whole Pyramid A B C D. And becauſe B F is equal to F C, the Parallelogram E B F G will be double to the Triangle G F C. And ſince there are two Priſms of equal Altitude, one of which has that Parallelogram for a Baſe, and the other the Triangle, and the Parallelogram is double to the Triangle; thoſe Priſms will be † equal to one another. Therefore the Priſm contained under the two Triangles B K F, E H G, and the three Parallelograms E B F G, E B K H, K H G F, is equal to the Priſm contained under the two Triangles G F C, H K L, and the three Parallelograms K F C L, L C G H, H K F G. And it is manifeſt, that each of thoſe Priſms, the Baſe of one of which is the Parallelogram E B G F, and oppoſite Baſe to that the Right Line † 40. 11.

R K H,

KH, and the Baſe of the other Triangle GFC, and the oppoſite Baſe to this, the Triangle KLH, are greater than either of the Pyramids, whoſe Baſes are the Triangles AEG, HKL, and Vertices the Points H and D. For ſince, if the Right Lines EF, EH, be joined, the Priſm, whoſe Baſe is the Parallelogram EBFG, and oppoſite Baſe to that the Right Line KH, is greater than the Pyramid, whoſe Baſe is the Triangle EBF, and Vertex the Point K. But the Pyramid whoſe Baſe is the Triangle EBF, and Vertex the Point K, is equal to the Pyramid whoſe Baſe is the Triangle AEG, and Vertex the Point H. For they are contained under equal and ſimilar Planes. Wherefore the Priſm whoſe Baſe is the Parallelogram EBFG, and oppoſite Baſe to it the Right Line HK, is greater than the Pyramid whoſe Baſe is the Triangle AEG, and Vertex the Point H. But the Priſm whoſe Baſe is the Parallelogram EBFG, and oppoſite Baſe to it the Right Line HK, is equal to the Priſm whoſe Baſe is the Triangle GFC, and oppoſite Baſe to this the Triangle HKL: And the Pyramid whoſe Baſe is the Triangle AEG, and Vertex the Point H, is equal to the Pyramid whoſe Baſe is the Triangle HKL, and Vertex the Point D. Therefore the two Priſms aforeſaid are greater than the ſaid two Pyramids, whoſe Baſes are the Triangles AEG, HKL, and Vertices the Points H, D. And ſo the whole Pyramid whoſe Baſe is the Triangle ABC, and Vertex the Point D, is divided into two equal Pyramids, ſimilar to each other, and to the Whole: And into two equal Priſms; which two Priſms together are greater than half of the whole Pyramid. Therefore, *Every Pyramid having a triangular Baſe may be divided into two Pyramids, equal and ſimilar to one another, having triangular Baſes, and ſimilar to the whole Pyramid, and into two equal Priſms, which two Priſms are greater than the Half of the whole Pyramid*; which was to be demonſtrated.

PRO-

PROPOSITION IV.

THEOREM.

If there are two Pyramids of the ſame Altitude, having triangular Baſes, and each of them be divided into two Pyramids, equal to one another, and ſimilar to the Whole, as alſo into two equal Priſms; and if in like manner each of the two Pyramids, made by the former Diviſion, be divided, and this be done continually; then, as the Baſe of one Pyramid is to the Baſe of the other Pyramid, ſo are all the Priſms that are in one Pyramid, to all the Priſms that are in the other Pyramid, being equal in Multitude.

LET there be two Pyramids of the ſame Altitude, having the triangular Baſes A B C, D E F, whoſe Vertices are the Points G, H; and let each of them be divided into two Pyramids, equal to one another, and ſimilar to the Whole, and into two equal Priſms; and if in like manner each of the Pyramids made by the former Diviſion be conceived to be divided, and this be done continually, I ſay, as the Baſe A B C is to the Baſe D E F, ſo are all the Priſms that are in the Pyramid A B C G to all the Priſms that are in the Pyramid D E F H, being equal in Multitude.

For, ſince B X is equal to X C, and A L to L C, X L ſhall be * parallel to A B, and the Triangle A B C * 2. 6.
ſimilar to the Triangle L X C. For the ſame Reaſon the Triangle D E F ſhall be alſo ſimilar to the Triangle R Q F. And becauſe B C is double to C X, and E F to F Q, it ſhall be as B C is to C X, ſo is E F to F Q. And ſince there are deſcribed upon B C, C X, Right-lined Figures A B C, L X C, ſimilar and alike ſituate, and upon E F, F Q, Right-lined Figures D E F, R Q F, ſimilar and alike ſituate; therefore, as the Triangle B A C is to the Triangle L X C, ſo is † the † 22. 6.
Triangle D E F to the Triangle R Q F; and (by Alternation) as the Triangle A B C is to the Triangle D E F, ſo is the Triangle L X C to the Triangle R Q F. But as the Triangle L X C is to the Triangle R Q F, ſo is ‡ the Priſm, whoſe Baſe is the Triangle L X C, ‡ 28. and
and oppoſite Baſe to that the Triagle O M N, to the 32. 11.

Priſm, whoſe Baſe is the Triangle R Q F, and oppoſite Baſe to that the Triangle S T Y. Therefore, as the Triangle A B C is to the Triangle D E F, ſo is* the Priſm whoſe Baſe is the Triangle L X C, and oppoſite Baſe to that the Triangle O M N, to the Priſm whoſe Baſe is the Triangle R Q F, and oppoſite Baſe to that the Triangle S T Y; and becauſe the two Priſms that are in the Pyramid A B C G are equal to one another, as alſo thoſe two that are in the Pyramid D E F H; it ſhall be as the Priſm whoſe Baſe is the Parallelogram K L X B, and oppoſite Baſe to that the Right Line M O, is to the Priſm whoſe Baſe is the Triangle L X C, and oppoſite Baſe to that the Triangle O M N, ſo is the Priſm whoſe Baſe is the Parallelogram EPRQ, and oppoſite Baſe to that the Right Line ST, to the Priſm whoſe Baſe is the Triangle R Q F, and oppoſite Baſe to that the Triangle S T Y. Therefore (by compounding) as the Priſms KBXLMO, LXCMNO, to the Priſm LXCMNO, ſo the Priſms PEQRST, RQFSTY, to the Priſm RQFSTY. And (by Alternation) as the Priſms KBXLMO, LXCMNO, to the Priſms PEQRST, RQFSTY, ſo the Priſm LXCMNO, to the Priſm RQFSTY; but as the Priſm LXCMNO is to the Priſm RQFSTY, ſo has the Baſe L X C been proved to be to the Baſe R F Q; and ſo the Baſe A B C to the Baſe D E F. Therefore alſo as the Triangle A B C is to the Triangle D E F, ſo are the two Priſms that are in the Pyramid A B C G, to the two Priſms that are in the Pyramid D E F H. If in the ſame manner each of the Pyramids O M N G, S T Y H, made by the former Diviſion, be divided, it ſhall be as the Baſe O M N is to the Baſe S T Y, ſo the two Priſms that are in the Pyramid O M N G, to the two Priſms that are in the Pyramid S T Y H. But as the Baſe OMN is to the Baſe STY, ſo is the Baſe A B C to the Baſe D E F. Therefore as the Baſe A B C is to the Baſe D E F, ſo are the two Priſms that are in the Pyramid A B C G, to the two Priſms that are in the Pyramid D E F H; and ſo the two Priſms that are in the Pyramid O M N G, to the two Priſms that are in the Pyramid S T Y H, and ſo the four to the four. We demonſtrate the ſame of Priſms made by the Diviſion of the Pyramids A K L O, D P R S, and

* 11. 5.

and of all other Priſms, being equal in Multitude; *which was to be demonſtrated.*

PROPOSITION V.

THEOREM.

Pyramids of the ſame Altitude, and having triangular Baſes, are to one another as their Baſes.

LET there be two Pyramids of the ſame Altitude, having the triangular Baſes A B C, D E F, whoſe Vertices are the Points G, H. I ſay, as the Baſe A B C is to the Baſe D E F, ſo is the Pyramid A B C G to the Pyramid D E F H.

For, if it be not ſo, then it ſhall be as the Baſe A B C is to the Baſe D E F, ſo is the Pyramid A B C G to ſome Solid, greater or leſs than the Pyramid D E F H. Firſt, let it be to a Solid leſs, which let be Z, and divide the Pyramid D E F H into two Pyramids equal to each other, and ſimilar to the Whole, and into two equal Priſms; then theſe two Priſms are greater than the Half of the whole Pyramid. And again, let the Pyramids made by the former Diviſion, be divided after the ſame manner, and let this be done continually, until the Pyramids in the Pyramid D E F H, are leſs than the Exceſs by which the Pyramid D E F H exceeds the Solid Z. Let theſe, for Example, be the Pyramids D P R S, S T Y H; then the Priſms remaining in the Pyramid D E F H, are greater than the Solid Z. Alſo, let the Pyramid A B C G be divided into the ſame Number of ſimilar Parts, as the Pyramid D E F H is; and then, as the Baſe A B C is to the Baſe D E F, ſo * the Priſms that are in the Pyramid A B C G, to the Priſms that are in the Pyramid D E F H. But as the Baſe A B C is to the Baſe D E F, ſo is the Pyramid A B C G to the Solid Z. And therefore, as the Pyramid A B C G is to the Solid Z, ſo are the Priſms that are in the Pyramid A B C G, to the Priſms that are in the Pyramid D E F H; but the Pyramid A B C G is greater than the Priſms that are in it. Wherefore alſo the Solid Z is greater than the Priſms that are in the Pyramid D E F H; but it

* 4. of this.

* *From what has been already demonſtrated.*

it is leſs * alſo, which is abſurd. Therefore the Baſe ABC to the Baſe DEF, is not as the Pyramid ABCG to ſome Solid leſs than the Pyramid DEFH. After the ſame manner we demonſtrate, that the Baſe DEF to the Baſe ABC, is not as the Pyramid DEFH to ſome Solid leſs than the Pyramid ABCG. Therefore, I ſay, neither is the Baſe ABC to the Baſe DEF, as the Pyramid ABCG to ſome Solid greater than the Pyramid DEFH. For if this is poſſible, let it be to the Solid I, greater than the Pyramid DEFH. Then (by Inverſion) the Baſe DEF ſhall be to the Baſe ABC, as the Solid I to the Pyramid ABCG: But ſince the Solid I is greater than the Pyramid EDFH, it ſhall be as the Solid I is to the Pyramid ABCG, ſo is the Pyramid DEFH, to ſome Solid leſs than the Pyramid ABCG, as juſt now has been proved. And ſo, as the Baſe DEF is to the Baſe ABC, ſo is the Pyramid DEFH, to ſome Solid leſs than the Pyramid ABCG, which is abſurd. Therefore the Baſe ABC to the Baſe DEF, is not as the Pyramid ABCG to ſome Solid greater than the Pyramid DEFH. But it has been alſo proved, that the Baſe ABC to the Baſe DEF, is not as the Pyramid ABCG to ſome Solid leſs than the Pyramid DEFH. Wherefore, as the Baſe ABC is to the Baſe DEF, ſo is the Pyramid ABCG to the Pyramid DEFH. Therefore, *Pyramids of the ſame Altitude, and having triangular Baſes, are to one another as their Baſes*; which was to be demonſtrated.

PROPOSITION VI.

THEOREM.

Pyramids of the ſame Altitude, and having polygonous Baſes, are to one another as their Baſes.

LET there be Pyramids of the ſame Altitude, which have the polygonous Baſes ABCDE, FGHKL, and let their Vertices be the Points M, N. I ſay, as the Baſe ABCDE is to the Baſe FGHKL, ſo is the Pyramid ABCDEM to the Pyramid FGHKLN.

For, let the Baſe ABCDE be divided into the Triangles ABC, ACD, ADE; and the Baſe FGHKL into the Triangles FGH, FHK, FKL; and let Pyramids be conceived upon every one of thoſe Triangles of the ſame Altitude with the Pyramids ABC, DEM, FGHKLN. Then, becauſe the Triangle ABC is to the Triangle ACD, as * the Pyramid ABCM is to the Pyramid ACDM: And (by compounding) as the Trapezium ABCD is to the Triangle ACD, ſo is the Pyramid ABCDM to the Pyramid ACDM; but as the Triangle ACD is to the Triangle ADE, ſo is * the Pyramid ACDM to the Pyramid ADEM. Wherefore, (by Equality of Proportion) as the Baſe ABCD is to the Baſe ADE, ſo is the Pyramid ABCDM to the Pyramid ADEM. And again (by Compoſition of Proportion) as the Baſe ABCDE is to the Baſe ADE, ſo is the Pyramid ABCDEM to the Pyramid ADEM. For the ſame Reaſon, as the Baſe FGHKL is to the Baſe FKL, ſo is the Pyramid FGHKLN to the Pyramid FKLN. And ſince there are two Pyramids ADEM, FKLN, having triangular Baſes, and the ſame Altitude, the Baſe ADE ſhall be * to the Baſe FKL, as the Pyramid ADEM to the Pyramid FKLN. And ſince the Baſe ABCDE is to the Baſe ADE, as the Pyramid ABCDEM is to the Pyramid ADEM; and as the Baſe ADE is to the Baſe FKL, ſo is the Pyramid ADEM to the Pyramid FKLN; it ſhall be (by Equality of Proportion) as the Baſe ABCDE to the Baſe FKL, ſo is the Pyramid ABCDEM to the Pyramid FKLN; but as the Baſe FKL is to the Baſe FGHKL, ſo was the Pyramid FKLN to the Pyramid FGHKLN. Wherefore, again, (by Equality of Proportion) as the Baſe ABCDE is to the Baſe FGHKL, ſo is the Pyramid ABCDEM to the Pyramid FGHKLN. Therefore, *Pyramids of the ſame Altitude, and having polygonous Baſes, are to one another as their Baſes*; which was to be demonſtrated.

* 5 *of this.*

PROPOSITION VII.

THEOREM.

Every Priſm, having a triangular Baſe, may be divided into three Pyramids equal to one another, and having triangular Baſes.

LET there be a Priſm whoſe Baſe is the Triangle ABC, and oppoſite Baſe to that the Triangle DEF. I ſay, the Priſm ABCDEF may be divided into the three equal Pyramids, that have triangular Baſes.

For, join BD, EC, CD. Then, becauſe ABED is a Parallelogram, whoſe Diameter is BD, the Triangle ABD ſhall be * equal to the Triangle EBD. Therefore the Pyramid whoſe Baſe is the Triangle ABD, and Vertex the Point C, is † equal to the Pyramid whoſe Baſe is the Triangle EDB, and Vertex the Point C. But the Pyramid whoſe Baſe is the Triangle EDB, and Vertex the Point C, is the ſame as the Pyramid whoſe Baſe is the Triangle EBC, and Vertex the Point D; for they are contained under the ſame Planes. Therefore the Pyramid, whoſe Baſe is the Triangle ABD, and Vertex the Point C, is equal to the Pyramid whoſe Baſe is the Triangle EBC, and Vertex the Point D. Again, becauſe FCBE is a Parallelogram, whoſe Diameter is CE, the Triangle ECF ſhall be * equal to the Triangle CBE. And ſo the Pyramid whoſe Baſe is the Triangle BEC, and Vertex the Point D, is † equal to the Pyramid, whoſe Baſe is the Triangle ECF, and Vertex the Point D: But the Pyramid whoſe Baſe is the Triangle BCE, and Vertex the Point D, has been proved equal to the Pyramid whoſe Baſe is the Triangle ABD, and Vertex the Point C. Wherefore alſo the Pyramid, whoſe Baſe is the Triangle CEF, and Vertex the Point D, is equal to the Pyramid, whoſe Baſe is the Triangle ABD, and Vertex the Point C. Therefore the Priſm ABCDEF is divided into three Pyramids equal to one another, and having triangular Baſes. And becauſe the Pyramid, whoſe Baſe is the Triangle ABD, and Vertex the Point C, is the ſame with the Pyramid whoſe Baſe is the Triangle CAB, and Vertex the Point D; for they

* 34. 1.

† 6 *of this.*

they are contained under the ſame Planes; and the Pyramid, whoſe Baſe is the Triangle A B D, and Vertex the Point C, has been proved to be a third Part of the Priſm, whoſe Baſe is the Triangle A B C, and oppoſite Baſe to that the Triangle D E F; therefore alſo the Pyramid, whoſe Baſe is the Triangle A B C, and Vertex the Point D, is a third Part of the Priſm having the ſame Baſe, *viz.* the Triangle A B C, and the oppoſite Baſe the Triangle D E F; *which was to be demonſtrated.*

Coroll. 1. It is manifeſt from hence, that every Pyramid is a third Part of a Priſm, having the ſame Baſe, and an equal Altitude; becauſe, if the Baſe of a Priſm, as alſo the oppoſite Baſe, be of any other Figure, it may be divided into Priſms having triangular Baſes.

2. Priſms of the ſame Altitude are to one another as their Baſes.

PROPOSITION VIII.

THEOREM.

Similar Pyramids, having triangular Baſes, are in a triplicate Proportion of their homologous Sides.

LET there be two Pyramids ſimilar and alike ſituate, having the triangular Baſes A B C, D E F, and let their Vertices be the Points G, H. I ſay, the Pyramid A B C G to the Pyramid D E F H has a Proportion triplicate of that which B C has to E F.

For, complete the ſolid Parallelepipedons B G M L, E H P O; then becauſe the Pyramid A B C G is ſimilar to the Pyramid D E F H, the Angle A B C ſhall be * equal to the Angle D E F, the Angle G B C equal to the Angle H E F, and the Angle A B G equal to the Angle D E H. And A B is to D E as B C is to E F; and ſo is B G to E H. Therefore becauſe A B is to D E, as B C is to E F; and the Sides about the equal Angles are proportional, the Parallelogram B M ſhall † be ſimilar to the Parallelogram E P. For the ſame Reaſon, the Parallelogram B N is ſimilar to the Parallelogram E R, and the Parallelogram

* *Def.* 9. 11.

† 6. 6.

rallelogram BK to the Parallelogram EX. Therefore three Parallelograms BM, KB, BN, are ſimilar to three Parallelograms EP, EX, ER; but the three MB, BK, BN, are equal and ſimilar to the three oppoſite ones; as alſo the three EP, EX, ER. Therefore the Solids BGML, EHPO, are contained under equal Numbers of ſimilar and equal Planes; and conſequently, the Solid BGML is ſimilar to the Solid EHPO. But ſimilar ſolid Parallelepipedons are * to each other in a triplicate Proportion of their homologous Sides. Therefore the Solid BGML to the Solid EHPO, has a Proportion triplicate of that which the homologous Side BC has to the homologous Side EF. But as the Solid BCML is to the Solid EHPO, ſo is † the Pyramid ABCG to the Pyramid DEFH; for the Pyramid is the one ſixth Part of that Solid, ſince the Priſm, which is the Half of the Solid Parallepipedon, is triple of the Pyramid. Wherefore the Pyramid ABCG to the Pyramid DEFH, ſhall have a triplicate Proportion to that which BC has to EF; *which was to be demonſtrated.*

* 33. 11.

† 15. 5.

Coroll. From hence it is manifeſt, that ſimilar Pyramids, having polygonous Baſes, are to one another in a triplicate Proportion of their homologous Sides. For, if they be divided into Pyramids having triangular Baſes; becauſe their ſimilar polygonous Baſes are divided into ſimilar Triangles equal in Number, and homologous to the Wholes, it ſhall be as one Pyramid having a triangular Baſe in one of the Pyramids, is to a Pyramid having a triangular Baſe in the other Pyramid, ſo are all the Pyramids having triangular Baſes in one Pyramid, to all the Pyramids having triangular Baſes in the other Pyramid; that is, ſo is one of the Pyramids having the polygonous Baſe, to the other; but a Pyramid having a triangular Baſe to a Pyramid having a triangular Baſe, is in a triplicate Proportion of the homologous Sides. Therefore one Pyramid having a polygonous Baſe to another Pyramid having a ſimilar Baſe, is in a triplicate Proportion of their homologous Sides.

PROPOSITION IX.

THEOREM.

The Baſes and Altitudes of equal Pyramids, having triangular Baſes, are reciprocally proportional; and thoſe Pyramids, having triangular Baſes, whoſe Baſes and Altitudes are reciprocally proportional, are equal.

LET there be equal Pyramids, having the triangular Baſes ABC, DEF, and Vertices the Points G, H. I ſay, the Baſes and Altitudes of the Pyramids ABCG, DEFH, are reciprocally proportional; that is, as the Baſe ABC is to the Baſe DEF, ſo is the Altitude of the Pyramid DEFH to the Altitude of the Pyramid ABCG.

For, complete the ſolid Parallelepipedons BGML, EHPO. Then, becauſe the Pyramid ABCG is equal to the Pyramid DEFH, and the Solid BGML is ſextuple, the Pyramid ABCG, and the Solid EHPO, ſextuple of the Solid DEFH, the Solid BGML ſhall be* equal to the Solid EHPO. But the Baſes and Altitudes of equal ſolid Parallelepipedons are reciprocally proportional. Therefore, as the Baſe BM is to the Baſe EP, ſo is† the Altitude of the Solid EHPO to the Altitude of the Solid BGML. But as the Baſe BM is to the Baſe EP, ſo is † the Triangle ABC to the Triangle DEF. Therefore, as the Triangle ABC is to the Triangle DEF, ſo is the Altitude of the Solid EHPO to the Altitude of the Solid BGML. But the Altitude of the Solid EHPO is the ſame as the Altitude of the Pyramid DEFH; and the Altitude of the Solid BGML the ſame as the Altitude of the Pyramid ABCG. Therefore, as the Baſe ABC is to the Baſe DEF, ſo is the Altitude of the Pyramid DEFH to the Altitude of the Pyramid ABCG. Wherefore the Baſes and Altitudes of the equal Pyramids ABCG, DEFH, are reciprocally proportional; and if the Baſes and Altitudes of the Pyramids ABCG, DEFH, are reciprocally proportional, that is, if the Baſe ABC to the Baſe DEF, be as the Altitude of the Pyramid DEFH

* 15. 5.

† 34. 11.

DEFH to the Altitude of the Pyramid ABCG; I ſay, the Pyramid ABCG is equal to the Pyramid DEFH: For, the ſame Conſtruction remaining, becauſe the Baſe ABC to the Baſe DEF, is as the Altitude of the Pyramid DEFH to the Altitude of the Pyramid ABCG; and as the Baſe ABC is to the Baſe DEF, ſo is the Parallelogram BM to the Parallelogram EP; the Parallelogram BM to the Parallelogram EP ſhall be alſo as the Altitude of the Pyramid DEFH is to the Altitude of the Pyramid ABCG. But as the Altitude of the Pyramid DEFH is the ſame as the Altitude of the ſolid Parallelepipedon EHPO, and the Altitude of the Pyramid ABCG the ſame as the Altitude of the ſolid Parallelepipedon BGML; therefore the Baſe BM to the Baſe EP will be as the Altitude of the ſolid Parallelepipedon EHPO to the Altitude of the ſolid Parallelepipedon BGML. But thoſe ſolid Parallelepipedons, whoſe Baſes and Altitudes are reciprocally proportional, are † equal to each other. Therefore the ſolid Parallelepipedon BGML is equal to the ſolid Parallelepipedon EHPO; and the Pyramid ABCG is a ſixth Part of the Solid BGML. And in like manner the Pyramid DEFH is a ſixth Part of the Solid EHPO. Therefore the Pyramid ABCG is equal to the Pyramid DEFH. Wherefore *the Baſes and Altitudes of equal Pyramids, having triangular Baſes, are reciprocally proportional; and thoſe Pyramids, having triangular Baſes, whoſe Baſes and Altitudes are reciprocally proportional, are equal*; which was to be demonſtrated.

† 34. 11.

PROPOSITION X.

THEOREM.

Every Cone is a third Part of a Cylinder, having the ſame Baſe, and an equal Altitude.

LET a Cone have the ſame Baſe as a Cylinder, *viz.* the Circle ABCD, and an Altitude equal to it. I ſay, the Cone is a third Part of the Cylinder; that is, the Cylinder is triple to the Cone.

For, if the Cylinder be not triple to the Cone, it ſhall be greater or leſs than triple thereof. Firſt, let it be greater than triple to the Cone, and let the Square A B C D be deſcribed in the Circle A B C D; then the Square A B C D is greater than one half of the Circle A B C D. Now let a Priſm be erected upon the Square A B C D, having the ſame Altitude as the Cylinder, and this Priſm will be greater than one Half of the Cylinder; becauſe, if a Square be circumſcribed about the Circle A B C D, the inſcribed Square will be one half of the circumſcribed Square; and if a Priſm be erected upon the circumſcribed Square of the ſame Altitude as the Cylinder, ſince Priſms are * to one another as their Baſes, the Priſm erected upon the Square A B C D is one Half of the Priſm erected upon the Square deſcribed about the Circle A B C D. But the Cylinder is leſſer than the Priſm erected on the Square deſcribed about the Circle A B C D. Therefore the Priſm erected on the Square A B C D, having the ſame Height as the Cylinder, is greater than one Half of the Cylinder. Let the Circumferences A B, B C, C D, D A, be biſected in the Points E, F, G, H; and join A E, E B, B F, F C, C G, G D, D H, H A. Then each of the Triangles A E B, B F C, C G D, D H A, is † greater than the Half of each of the Segments in which they ſtand. Let Priſms be erected from each of the Triangles A E B, B F C, C G D, D H A, of the ſame Altitude as the Cylinder; then every one of theſe Priſms erected is greater than its correſpondent Segment of the Cylinder. For, becauſe, if Parallels be drawn through the Points E, F, G, H, to A B, B C, C D, D A, and Parallelograms be completed on the ſaid A B, B C, C D, D A, on which are erected ſolid Parallelepipedons of the ſame Altitude as the Cylinder; then each of thoſe Priſms that are on the Triangles A E B, B F C, C G D, D H A, are Halves † of each of the ſolid Parallelepipedons; and the Segments of the Cylinder are leſs than the erected ſolid Parallelepipedons; and conſequently the Priſms that are on the Triangles A E B, B F C, C G D, D H A, are greater than the Halves of the Segments of the Cylinder; and ſo biſecting the other Circumferences, joining Right Lines, and on every one of the Triangles erecting Priſms of the ſame Height as the Cylinder;

* *2 Cor. 7. of this.*

† *This follows from 2. of this.*

Cylinder; and doing this continually, we ſhall at laſt have certain Portions of the Cylinder left, that are leſs than the Exceſs by which the Cylinder exceeds triple the Cone.

Now let theſe Portions remaining be AE, EB, BF, FC, CG, GD, DH, HA. Then the Priſm remaining, whoſe Baſe is the Polygon AEBFCGDH, and Altitude equal to that of the Cylinders, is greater than the Triple of the Cone. But the Priſm whoſe Baſe is the Polygon AEBFCGDH, and Altitude the ſame as that of the Cylinder's, is* triple of the Pyramid whoſe Baſe is the Polygon AEBFCGDH, and Vertex the ſame as that of the Cone. And therefore the Pyramid whoſe Baſe is the Polygon AEBFCGDH, and Vertex the ſame as that of the Cone, is greater than the Cone whoſe Baſe is the Circle ABCD; but it is leſſer alſo (for it is comprehended by it); which is abſurd. Therefore the Cylinder is not greater than triple the Cone. I ſay, it is neither leſſer than triple the Cone: For if it be poſſible, let the Cylinder be leſs than triple the Cone: Then (by Inverſion) the Cone ſhall be greater than a third Part of the Cylinder. Let the Square ABCD be deſcribed in the Circle ABCD; then the Square ABCD is greater than half of the Circle ABCD. And let a Pyramid be erected on the Square ABCD having the ſame Vertex as the Cone, then the Pyramid erected is greater than one Half of the Cone; becauſe, as has been already demonſtrated, if a Square be deſcribed about the Circle, the Square ABCD ſhall be half thereof. And if ſolid Parallelepipedons be erected upon the Squares of the ſame Altitude as the Cones, which are alſo called Priſms; then the Priſm erected on the Square ABCD is one Half of that erected on the Square deſcribed about the Circle; for they are to each other as their Baſes, and ſo likewiſe are their third Parts. Therefore the Pyramid whoſe Baſe is the Square ABCD, is one Half of that Pyramid erected upon the Square deſcribed about the Circle: But the Pyramid erected upon the Square deſcribed about the Circle, is greater than the Cone; for it comprehends it. Therefore the Pyramid whoſe Baſe is the Square ABCD, and Vertex the ſame as that of the Cone, is greater than one Half of

* 1 *Cor.* 7. *of this.*

of the Cone. Biſect the Circumferences AB, BC, CD, DA, in the Points E, F, G, H; and join AE, EB, BF, FC, CG, GD, DH, HA; and then each of the Triangles AEB, BFC, CGD, DHA, is greater than one Half of each of the Segments they are in. Let Pyramids be erected upon each of the Triangles AEB, BFC, CGD, DHA, having the ſame Vertex as the Cone: then each of theſe Pyramids thus erected, is greater than one Half of the Segment of the Cone in which it is: And ſo biſecting the remaining Circumferences, joining the Right Lines, and erecting Pyramids upon every of the Triangles having the ſame Altitude as the Cone, and doing this continually, we ſhall at laſt have Segments of the Cone left, that will be leſs than the Exceſs by which the Cone exceeds the one third Part of the Cylinder. Let theſe Segments be thoſe that are on AE, EB, BF, FC, CG, GD, DH, HA; and then the remaining Pyramid, whoſe Baſe is the Polygon AEBFCGDH, and Vertex the ſame as that of the Cone, is greater than a third Part of the Cylinder; but the Pyramid whoſe Baſe is the Polygon AEBFCGDH, and Vertex the ſame as that of the Cone, is one third Part of the Priſm whoſe Baſe is the Polygon AEBFCGDH, and Altitude the ſame as that of the Cylinder. Therefore the Priſm, whoſe Baſe is the Polygon AEBFCGDH, and Altitude the ſame as that of the Cylinder, is greater than the Cylinder, whoſe Baſe is the Circle ABCD; but it is leſs alſo, (as being comprehended thereby) which is abſurd; therefore the Cylinder is not leſs than triple of the Cone; but it has been proved alſo not to be greater than triple of the Cone; therefore the Cylinder is neceſſarily triple of the Cone. Wherefore, *Every Cone is a third Part of a Cylinder, having the ſame Baſe, and an equal Altitude*; which was to be demonſtrated.

PRO-

PROPOSITION XI.

THEOREM.

Cones and Cylinders of the ſame Altitude are to one another as their Baſes.

LET there be Cones and Cylinders of the ſame Altitude, whoſe Baſes are the Circles ABCD, EFGH, Axes KL, MN, and Diameters of the Baſes AC, EG. I ſay, as the Circle ABCD is to the Circle EFGH, ſo is the Cone AL to the Cone EN.

For if it be not ſo, it ſhall be as the Circle ABCD is to the Circle EFGH, ſo is the Cone AL to ſome Solid either leſs or greater than the Cone EN. Firſt, let it be to the Solid X leſs than the Cone; and let the Solid I be equal to the Exceſs of the Cone EN above the Solid X. Then the Cone EN is equal to the Solids X, I; let the Square EFGH be deſcribed in the Circle EFGH, which Square is greater than one Half of the Circle, and erect a Pyramid upon the Square EFGH of the ſame Altitude as the Cone. Therefore the Pyramid erected is greater than one Half of the Cone: For if we deſcribe a Square about the Circle, and a Pyramid be erected thereon of the ſame Altitude as the Cone, the Pyramid inſcribed will be one Half of the Pyramid circumſcribed; for they are * to one another as their Baſes, and the Cone is leſs than the circumſcribed Pyramid. Therefore the Pyramid whoſe Baſe is the Square EFGH, and Vertex the ſame as that of the Cone, is greater than one Half of the Cone. Biſect the Circumferences EF, FG, GH, HE, in the Points P, R, S, O, and join HO, OE, EP, PF, FR, RG, GS, SH; then each of the Triangles HOE, EPF, FRG, GHS, is greater than one half of the Segment of the Circle wherein it is. Let a Pyramid be raiſed upon every one of the Triangles HOE, EPF, FRG, GHS, of the ſame Altitude as the Cone. Then each of thoſe erected Pyramids is greater than one Half of its correſpondent Segment of the Cone: And ſo biſecting the remaining Circumferences, joining the Right Lines, and erecting Pyramids upon each of the Triangles of the

* *6 of this.*

the ſame Altitude as that of the Cone; and doing this continually, there will at laſt be left Segments of the Cone, that will together be leſs than the Solid I. Let thoſe be the Segments that are on HO, OE, EP, PF, FR, RG, GS, SH. Therefore the Pyramid remaining, whoſe Baſe is the Polygon HOEPFRGS, and Altitude the ſame as that of the Cone, is greater than the Solid X. Let the Polygon DTAYBQCV be deſcribed in the Circle ABCD, ſimilar and alike ſituate to the Polygon HOEPFRGS; and let a Pyramid be erected thereon of the ſame Altitude as the Cone AL. Then, becauſe the Square of AC to the Square of EG, is* as the Polygon DTAYBQCV to the Polygon HOEPFRGS; and the Square of AC is† to the Square of EG, as the Circle ABCD to the Circle EFGH; it ſhall be as the Circle ABCD to the Circle EFOH, ſo is the Polygon DTAYBQCV to the Polygon HOEPFRGS: But as the Circle ABCD is to the Circle EFGH, ſo is the Cone AL to the Solid X; and as the Polygon DTAYBQCV is to the Polygon HOEPFRGS, ſo is the Pyramid whoſe Baſe is the Polygon DTAYBQCV, and Vertex the Point L, to the Pyramid whoſe Baſe is the Polygon HOEPFRGS, and Vertex the Point N. Therefore, as the Cone AL to the Solid X, ſo the Pyramid whoſe Baſe is the Polygon DTAYBQCV, and Vertex the Point L, to the Pyramid whoſe Baſe is the Polygon HOEPFRGS, and Vertex the Point N; but the Cone AL is greater than the Pyramid that is in it. Therefore the Solid X is greater than the Pyramid that is in the Cone EN; but it was put leſs, which is abſurd. Therefore the Circle ABCD to the Circle EFGH, is not as the Cone AL to ſome Solid leſs than the Cone EN. In like manner, it is demonſtrated, that the Circle EFGH to the Circle ABCD, is not the Cone EN to ſome Solid leſs than the Cone AL. I ſay, moreover, that the Circle ABCD to the Circle EFGH, is not as the Cone AL to ſome Solid greater than the Cone EN: For, if it be poſſible, let it be to the Solid Z greater than the Cone; then, (by Inverſion) as the Circle EFGH is to the Circle ABCD, ſo ſhall the Solid Z be to the Cone AL. But ſince the Solid Z is greater than the Cone EN, it ſhall be as the Solid

* 1 *of this.*

† 2 *of this.*

Z is to the Cone A L, ſo is the Cone E N to ſome Solid leſs than the Cone A L. And therefore, as the Circle E F G H is to the Circle A B C D, ſo is the Cone E N to ſome Solid leſs than the Cone A L; which has been proved to be impoſſible. Therefore the Circle A B C D to the Circle E F G H, is not as the Cone A L to ſome Solid greater than the Cone E N. It has alſo been proved, that the Circle A B C D to the Circle E F G H, is not as the Cone A L to ſome Solid leſs than the Cone E N. Therefore as the Circle A B C D is to the Circle E F G H, ſo is the Cone A L to the Cone E N: But as Cone is to Cone, ſo is* Cylinder to Cylinder; for each Cylinder is triple of each Cone; and therefore, as the Circle A B C D is to the Circle E F G H, ſo are Cylinders and Cones ſtanding on them, of the ſame Altitude. Wherefore *Cones and Cylinders of the ſame Altitude, are to one another as their Baſes*; which was to be demonſtrated.

*15. 5.

PROPOSITION XII.

THEOREM.

Similar Cones and Cylinders are to one another in a triplicate Proportion of the Diameters of their Baſes.

LET there be ſimilar Cones and Cylinders, whoſe Baſes are the Circles A B C D, E F G H, and Diameters of the Baſes B D, F H, and Axes of the Cones or Cylinders K L, M N. I ſay, the Cone whoſe Baſe is the Circle A B C D, and Vertex the Point L, to the Cone whoſe Baſe is the Circle E F G H, and Vertex the Point N, hath a triplicate Proportion of that which B D has to F H.

For if the Cone A B C D L to the Cone E F G H N, has not a triplicate Proportion of that which B D has to F H, the Cone A B C D L ſhall have that triplicate Proportion to ſome Solid, either leſs or greater than the Cone E F G H N. Firſt, let it have that triplicate Proportion to the Solid X, leſs than the Cone E F G H N; and let the Square E F G H be deſcribed in the Circle E F G H, which will be greater than one Half of the Circle E F G H; and erect a Pyramid on the

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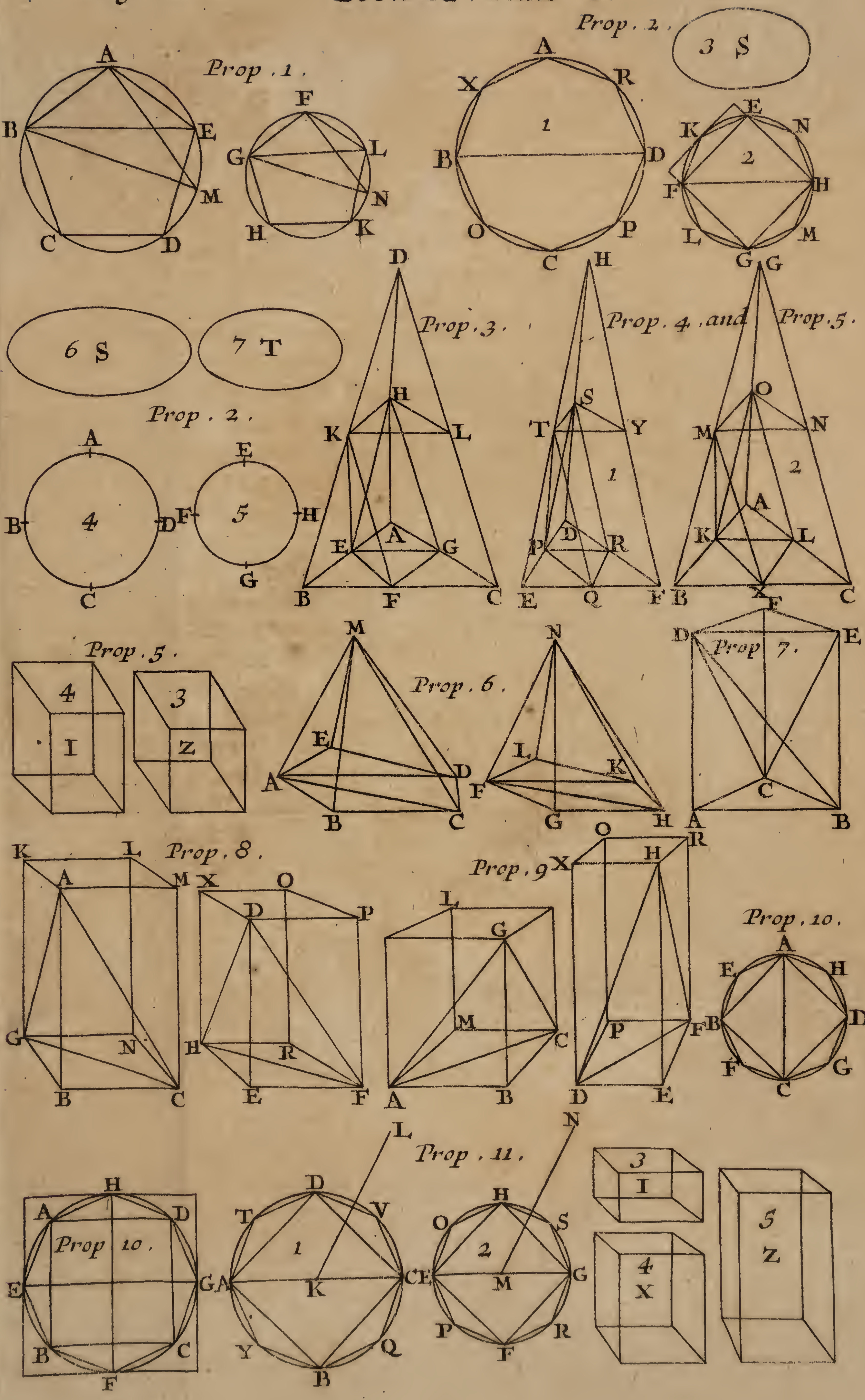
Book 12. Plate 1.



the Square EFGH of the ſame Altitude with the Cone, then that Pyramid is greater than one Half of the Cone. And ſo let the Circumferences EF, FG, GH, HE, be biſected in the Points O, P, R, S; and join EO, OF, FP, PG, GR, RH, HS, SE; then each of the Triangles EOF, FPG, GRH, HSE, is greater than one Half of the Segment of the Circle EFGH, in which it is; and erect a Pyramid upon each of the Triangles EOF, EPG, GRH, HSE, having the ſame Altitude as the Cone: Then each of the Pyramids thus erected, is greater than half its correſponding Segment of the Cone. Wherefore, biſecting the remaining Circumferences, joining Right Lines, and erecting Pyramids upon each of the Triangles, having the ſame Vertex as the Cone; and doing this continually, we ſhall leave at laſt certain Segments of the Cone, that ſhall be leſs than the Exceſs by which the Cone EFGHN exceeds the Solid X. Let theſe be the Segments that ſtand on EO, OF, FP, PG, GR, RH, HS, SE; then the remaining Pyramid, whoſe Baſe is the Polygon EOFPGRHS, and Vertex the Point N, is greater than the Solid X. Alſo let the Polygon ATBYCVDQ be deſcribed in the Circle ABCD, ſimilar and alike ſituate to the Polygon EOFPGRHS; upon which erect a Pyramid having the ſame Altitude as the Cone; and let LBT be one of the Triangles containing the Pyramid, whoſe Baſe is the Polygon ATBYCVDQ, and Vertex the Point L; as likewiſe NFO one of the Triangles containing the Pyramid EOFPGRHS, and Vertex the Point N; and let KT, MO, be join'd. Then, becauſe the Cone ABCDL is ſimilar to the Cone EFGHN, it ſhall be as BD is to FH, ſo is the Axis KL to the Axis MN; but as BD is to FH, ſo is* BK to FM; and as BK is to FM, conſequently ſo is KL to MN; and (by Alternation) as BK is to KL, ſo is FM to MN. And ſince each is perpendicular, and the Sides about the equal Angles BKL, FMN, are proportional, the Triangle BKL ſhall be† ſimilar to the Triangle FMN. Again, becauſe BK is to KT, as FM is to MO, the Sides are proportional about equal Angles BKT, FMO; for the Angle BKT is the ſame Part of the four Right Angles at the Centre K, as the Angle FMO is of the

* 15. 5.

† 6. 6.

four Right Angles at the Centre M: the Triangle BKT, ſhall be * ſimilar to the Triangle FMO; and becauſe it has been proved, that BK is to KL, as FM is to MN, and BK is equal to KT, and FM to MO, it ſhall be as TK is to KL, ſo is OM to MN: and the proportional Sides are about equal Angles TKL, OMN; for they are Right Angles. Therefore the Triangle LKT ſhall be ſimilar to the Triangle MNO. And ſince, by the Similarity of the Triangles BKL, FMN, it is as LB is to BK, ſo is NF to FM; and, by the Similarity of the Triangles BKT, FMO, it is as KB is to BT, ſo is MF to FO; it ſhall be (by Equality of Proportion) as LB is to BT, ſo is NF to FO. Again, ſince by the Similarity of the Triangles LTK, NOM, it is as LT is to TK, ſo is NO to OM; and, by the Similarity of the Triangles KBT, OMF, it is as KT is to TB, ſo is MO to OF. It ſhall be (by Equality of Proportion) as LT is to TB, ſo is NO to OF: But it has been proved, that TB is to BL, as OF is to FN. Wherefore, again, (by Equality of Proportion) as TL is to LB, ſo is ON to NF; and therefore the Sides of the Triangles LTB, NOF, are proportional; and ſo the Triangles LTB, NOF, are equiangular and ſimilar to each other. And conſequently the Pyramid, whoſe Baſe is the Triangle BKT, and Vertex the Point L, is ſimilar to the Pyramid whoſe Baſe is the Triangle FMO, and Vertex the Point N; for they are contained under ſimilar Planes equal in Multitude: But ſimilar Pyramids, that have triangular Baſes, are † to one another in the triplicate Proportion of their homologous Sides. Therefore the Pyramid BKTL to the Pyramid FMON has a triplicate Proportion of that which BK has to FM. In like manner, drawing Right Lines from the Points A, Q, D, V, C, Y, to K, as alſo others, from the Points E, S, H, R, G, P, to M, and erecting Pyramids on the Triangles having the ſame Vertices as the Cones, we demonſtrate that every Pyramid of one Cone, to every one of the other Cone, has a triplicate Proportion of that which the Side BK has to the homologous Side MF, that is, which BD has to FH. But as one of the Antecedents is to one of the Conſequents, ſo are ‡ all the Antecedents to all the Conſequents Therefore, as the

* 6. 6.

† 8 of this.

‡ 12. 5.

the Pyramid BKTL is to the Pyramid FMON, ſo is the whole Pyramid whoſe Baſe is the Polygon ATBYCVDQ, and Vertex the Point L, to the whole Pyramid, whoſe Baſe is the Polygon EOFPGRHS, and Vertex the Point N. Wherefore the Pyramid, whoſe Baſe is the Polygon ATBYCVDQ, and Vertex the Point L, to the Pyramid whoſe Baſe is the Polygon EOFPGRHS, and Vertex the Point N, has a triplicate Proportion of that which BD hath to FH. But the Cone whoſe Baſe is the Circle ABCD, and Vertex the Point L, is ſuppoſed to have to the Solid X a triplicate Proportion of that which BD has to FH. Therefore, as the Cone, whoſe Baſe is the Circle ABCD, and Vertex the Point L, is to the Solid X, ſo is the Pyramid whoſe Baſe is the Polygon ATBYCVDQ, and Vertex the Point L, to the Pyramid whoſe Baſe is the Polygon EOFPGRHS, and Vertex the Point N. But the ſaid Cone is greater than the Pyramid that is in it; for it comprehends it. Therefore the Solid X alſo is greater than the Pyramid, whoſe Baſe is the Polygon EOFPGRHS, and Vertex the Point N; but it is alſo leſs, which is abſurd. Therefore, the Cone, whoſe Baſe is the Circle ABCD, and Vertex the Point L, to ſome Solid leſs than the Cone, whoſe Baſe is the Circle EFGH, and Vertex the Point N, has not a triplicate Proportion of that which BD has to FH. In like manner, we demonſtrate that the Cone EFGHN, to ſome Solid leſs than the Cone ABCDL, has not a triplicate Proportion of that which FH has to BD. Laſtly, I ſay, the Cone ABCDL, to a Solid greater than the Cone EFGHN, has not a triplicate Proportion of that which BD has to FH: For, if this be poſſible, let it be ſo to ſome ſolid Z greater than the Cone EFGHN. Then (by Inverſion) the Solid Z, to the Cone ABCDL, has a triplicate Proportion of that which FH has to BD. But ſince the Solid Z is greater than the Cone EFGHN, the Solid Z ſhall be to the Cone ABCDL, as the Cone EFGHN is to ſome Solid leſs than the Cone ABCDL; and therefore the Cone EFGHN, to ſome Solid leſs than the Cone ABCDL, hath a triplicate Proportion of that which FH has to BD, which has been proved to be impoſſible. Therefore the Cone ABCDL, to

ſome

ſome Solid greater than the Cone EFGHN, has not a triplicate Proportion of that which BD has to FH. It has been alſo demonſtrated, that the Cone ABCDL, to ſome Solid leſs than the Cone EFGHN, hath not a triplicate Proportion of that which BD has to FH. Wherefore the Cone ABCDL, to the Cone EFGHN, has a triplicate Proportion of that which BD has to FH. * 15. 5. But as Cone is to Cone, ſo is * Cylinder to Cylinder. For a Cylinder having the ſame Baſe as a † 10 *of this.* Cone, and the ſame Altitude, is † triple of the Cone, ſince it is demonſtrated, that every Cone is one third Part of a Cylinder, having the ſame Baſe, and equal Altitude. Therefore alſo a Cylinder to Cylinder has a triplicate Proportion of that which BD has to FH. Therefore *ſimilar Cones and Cylinders are to one another in a triplicate Proportion of the Diameters of their Baſes*; which was to be demonſtrated.

PROPOSITION XIII.

THEOREM.

If a Cylinder be divided by a Plane parallel to the oppoſite Planes, then as one Cylinder is to the other Cylinder, ſo is the Axis to the Axis.

LET the Cylinder AD be divided by the Plane GH, parallel to the oppoſite Planes AB, CD, and meeting the Axis EF in the Point K. I ſay, As the Cylinder BG is to the Cylinder GD, ſo is the Axis EK to the Axis KF.

For, let the Axis EF be both ways produced to L and M, and put any Number of EN, NL, *&c.* each equal to the Axis EK; and any Number of FX, XM, *&c.* each equal to FK. And thro' the Points L, N, X, M, let Planes parallel to AB, CD, paſs. And in thoſe Planes from L, N, X, M, as Centres, deſcribe the Circles, OP, RS, TY, VQ, each equal to AB, CD; and conceive the Cylinders PR, RB, DT, TQ, to be completed. Then, becauſe the Axes LN, NE, EK, are equal to each other, the * 11 *of this.* Cylinders PR, RB, BG, will be * to one another as their Baſes. And therefore the Cylinders PR, RB, BG, are equal. And ſince the Axes LN, NE, EK, are equal to each other, as alſo the Cylinders P R, R B, B G; and the Number of L N, N E, E K, is equal to the Number of P R, R B, B G: The Axis K L ſhall be the ſame Multiple of the Axis E K, as the Cylinder P G is of the Cylinder G B. For the ſame Reaſon, the Axis M K is the ſame Multiple of the Axis K F, as the Cylinder G Q is of the Cylinder G D. Now, if the Axis K L be equal to the Axis K M, the Cylinder P G ſhall be equal to the Cylinder G Q; if the Axis L K be greater than the Axis K M, the Cylinder P G ſhall be likewiſe greater than the Cylinder G Q; and if leſs, leſs. Therefore, becauſe there are four Magnitudes, *viz.* the Axes E K, K F, and the Cylinders B G, G D; and there are taken their Equimultiples, namely, the Axis K L, and the Cylinder P G, the Equimultiples of the Axis E K, and the Cylinder B G; and the Axis K M, and the Cylinder G Q, the Equimultiples of the Axis K F, and the Cylinder G D: And it is demonſtrated, that if the Axis L K exceeds the Axis K M, the Cylinder P G will exceed the Cylinder G Q; and if it be equal, equal; and leſs, leſs. Therefore, as the Axis E K is to the Axis K F, ſo * is the Cylinder B G to the Cylinder G D. Wherefore, *if a Cylinder be divided by a Plane parallel to the oppoſite Planes, then as one Cylinder is to the other Cylinder, ſo is the Axis to the Axis*; which was to be demonſtrated

*Def. 5. 5.

PROPOSITION XIV.

THEOREM.

Cones and Cylinders, being upon equal Baſes, are to one another as their Altitudes.

LET the Cylinders E B, F D, ſtand upon equal Baſes A B, C D. I ſay, as the Cylinder E B is to the Cylinder F D, ſo is the Axis G H to the Axis K L.

For produce the Axis K L to the Point N; and put L N, equal to the Axis G H; and let a Cylinder C M be conceived about the Axis L N. Then, becauſe the Cylinders E B, C M, have the ſame Altitude, they are * to one another as their Baſes. But their

* 11 of this.

their Baſes are equal. Therefore the Cylinders EB, CM, will be alſo equal. And becauſe the Cylinder FM is cut by a Plane CD, parallel to the oppoſite Planes, it ſhall be as the Cylinder CM is to the Cylinder FD, ſo is the Axis LN to the Axis KL. But the Cylinder CM is equal to the Cylinder EB; and the Axis LN to the Axis GH. Therefore the Cylinder EB is to the Cylinder FD, as the Axis GH is to the Axis KL. And as the Cylinder EB is to
‡ 15. 5. the Cylinder FD, ſo is ‡ the Cone ABG to the Cone
‡ 10 *of this.* CDK; for the Cylinders are * triple of the Cones. Therefore, as the Axis GH is to the Axis KL, ſo is the Cone ABG to the Cone CDK, and ſo the Cylinder EB to the Cylinder FD. Wherefore *Cones and Cylinders, being upon equal Baſes, are to one another as their Altitudes*; which was to be demonſtrated.

PROPOSITION XV.

THEOREM.

The Baſes and Altitudes of equal Cones and Cylinders are reciprocally proportional; and Cones and Cylinders, whoſe Baſes and Altitudes are reciprocally proportional, are equal to one another.

LET the Baſes of the equal Cones and Cylinders be the Circles ABCD, EFGH, and their Diameters AC, EG; and Axes KL, MN; which are alſo the Altitudes of the Cones and Cylinders: And let the Cylinders AX, EO, be completed. I ſay, the Baſes and Altitudes of the Cylinders AX, EO, are reciprocally proportional; that is, the Baſe ABCD is to the Baſe EFGH, as the Altitude MN is to the Altitude KL.

For the Altitude KL is either equal to the Altitude MN, or not equal. Firſt, let it be equal; and the Cylinder AX is equal to the Cylinder EO. But Cylinders and Cones that have the ſame Altitude,
* 11 *of this.* are * to one another as their Baſes. Therefore the Baſe ABCD is equal to the Baſe EFGH. And conſequently, as the Baſe ABCD is to the Baſe EFGH, ſo is the Altitude MN to the Altitude KL.

But if the Altitude KL be not equal to the Altitude MN, let MN be the greater. And take PM equal to LK from MN; and let the Cylinder EO be cut thro' P by the Plane TYS, parallel to the oppoſite Planes of the Circles EFGH, RO; and conceive ES to be a Cylinder, whoſe Baſe is the Circle EFGH, and Altitude PM. Then, becauſe the Cylinder AX is equal to the Cylinder EO, and ES is ſome other Cylinder, the Cylinder AX to the Cylinder ES, ſhall be as the Cylinder EO is to the Cylinder ES. But as the Cylinder AX is to the Cylinder ES, ſo is * the Baſe ABCD to the Baſe EFGH; for the Cylinders AX, ES, have the ſame Altitude. And as the Cylinder EO is to the Cylinder ES, ſo is † the Altitude MN to the Altitude MP; for the Cylinder EO is cut by the Plane TYS parallel to the oppoſite Planes. Therefore, as the Baſe ABCD is to the Baſe EFGH, ſo is the Altitude MN to the Altitude MP. But the Altitude MP is equal to the Altitude KL. Wherefore, as the Baſe ABCD is to the Baſe EFGH, ſo is the Altitude MN to the Altitude KL. And therefore the Baſes and Altitudes of the equal Cylinders AX, EO, are reciprocally proportional.

* 11 *of this.*

† 13 *of this.*

And if the Baſes and Altitudes of the Cylinders AX, EO, are reciprocally proportional, that is, if the Baſe ABCD be to the Baſe EFGH, as the Altitude MN is to the Altitude KL; I ſay, the Cylinder AX is equal to the Cylinder EO. For, the ſame Conſtruction remaining; becauſe the Baſe ABCD is to the Baſe EFGH, as the Altitude MN is to the Altitude KL; and the Altitude KL is equal to the Altitude MP; it ſhall be as the Baſe ABCD is to the Baſe EFGH, ſo is the Altitude MN to the Altitude MP. But as the Baſe ABCD is to the Baſe EFGH, ſo is the Cylinder AX to the Cylinder ES; for they have the ſame Altitude. And as the Altitude MN is to the Altitude MP, ſo is ‡ the Cylinder EO to the Cylinder ES. Therefore, as the Cylinder AX is to the Cylinder ES, ſo is the Cylinder EO to the Cylinder ES. Wherefore the Cylinder AX is equal to the Cylinder EO. In like manner we prove this in Cones; *which was to be demonſtrated.*

‡ 11 *of this.*

PRO-

PROPOSITION XVI.

PROBLEM.

Two Circles being about the ſame Centre, to inſcribe in the greater a Polygon of equal Sides even in Number, that ſhall not touch the leſſer Circle.

LET ABCD, EFGH, be two given Circles about the Centre K. It is required to inſcribe a Polygon of equal Sides even in Number in the Circle ABCD, not touching the leſſer Circle EFGH.

Draw the Right Line BD through the Centre K, as alſo AG from the Point G at Right Angles to
* 16. 3. BD, which produce to C; this Line will * touch the Circle EFGH. Then biſecting the Circumference BAD, and again biſecting the Half thereof, and doing this continually, we ſhall have a Circumference left at laſt leſs than AD. Let this Circumference be LD, and draw LM from the Point L perpendicular to BD, which produce to N; and join LD, DN.
† 29. 3. And then LD is † equal to DN. And ſince LN is parallel to AC, and AC touches the Circle EFGH, LN will not touch the Circle EFGH. And much leſs do the Right Lines LD, DN, touch the Circle. And if Right Lines, each equal to LD, be applied round the Circle ABCD, we ſhall have a Polygon inſcribed therein of equal Sides, even in Number, that does not touch the leſſer Circle EFG; *which was to be demonſtrated.*

PROPOSITION XVII.

PROBLEM.

To deſcribe a ſolid Polyhedron, in the greater of two Spheres, having the ſame Centre, which ſhall not touch the Superficies of the leſſer Sphere.

LET two Spheres be ſuppoſed about the ſame Centre A. It is required to deſcribe a ſolid Polyhedron in the greater Sphere, not touching the Superficies of the leſſer Sphere.

Let the Spheres be cut by ſome Plane paſſing thro' the Centre. Then the Sections will be Circles; for becauſe a Sphere is * made by the turning of a Semicircle about the Diameter, which is at Reſt: In whatſoever Poſition the Semicircle is conceived to be, the Plane in which it is ſhall make a Circle in the Superficies of the Sphere. It is alſo manifeſt, that this Circle is a great Circle, ſince the Diameter of the Sphere, which is likewiſe the Diameter of the Semicircle, is † greater than all Right Lines that are drawn in the Circle or Sphere. Now, let B C D E be that Circle of the greater Sphere, and F G H of the leſſer Sphere; and let B D, C E, be two of their Diameters drawn at Right Angles to one another. Let B D meet the leſſer Circle in the Point G, from which to A G let G L be drawn at Right Angles, and A L joined. Then, biſecting the Circumference E B, as alſo the Half thereof, and doing thus continually, we ſhall have left at laſt a certain Circumference leſs than that Part of the Circumference of the Circle B C D, which is ſubtended by a Right Line equal to G L. Let this be the Circumference B K. Then the Right Line B K is leſs than G L; and B K ſhall be the Side of a Polygon of equal Sides, even in Number, not touching the leſſer Circle. Now, let the Sides of the Polygon in the Quadrant of the Circle B E, be the Right Lines B K, K L, L M, M E; and produce the Line joining the Points K, A, to N: And raiſe ‡ A X from the Point A, perpendicular to the Plane of the Circle B C D E, meeting the Superficies of the Sphere in the Point X; and let Planes be drawn thro' A X, and B D, and thro' A X, and K N, which, from what has been ſaid, will make great Circles in the Superficies of the Sphere. And let B X D, K X N, be Semicircles on the Diameters B D, K N. Then, becauſe X A is perpendicular to the Plane of the Circle B C D E, all Planes that paſs thro' X A ſhall alſo * be perpendicular to that ſame Plane. Therefore the Semicircles B X D, K X N, are perpendicular to that ſame Plane. And becauſe the Semicircles B E D, B X D, K X N, are equal; for they ſtand upon equal Diameters B D, K N; their Quadrants B E, B X, K X, ſhall be alſo equal. And therefore, as many Sides as the Polygon in the Quadrant B E has, ſo many Sides may there be

* *Def.* 14. 11.

† 15. 3.

‡ 12. 11.

* 18. 11.

be in the Quadrants BX, KX, equal to the Sides BK, KL, LM, ME. Let thoſe Sides be BO, OP, PR, RK, KS, ST, TY, YX: And join SO, TP, YR; and let Perpendiculars be drawn from O, S, to the Plane of the Circle BCDE. † 38. 11. Theſe will † fall on BD, KN, the common Sections of the Planes; becauſe the Planes of the Semicircles BXD, KXN, are perpendicular to the Plane of the Circle BCDE. Let the ſaid Perpendiculars be OV, SQ, and join VQ. Then, ſince the equal Circumferences BO, SK, are taken in the equal Semicircles BXD, KXN, and OV, SQ, are Perpendiculars, OV ſhall be equal to SQ, and BV to KQ. But the Whole BA is equal to the Whole KA. Therefore the Part remaining VA is equal to the Part remaining QA. Therefore, as BV is to VA, ſo is KQ ‡ 2. 6. to QA: And ſo VQ is ‡ parallel to BK. And ſince OV and SQ are both perpendicular to the Plane of * 6. 11. the Circle BCDE, OV ſhall be * parallel to SQ. But it has alſo been proved equal to it. Wherefore † 33. 1. QV, SO, are † equal and parallel. And becauſe QV is parallel to SO, and alſo parallel to KB, SO ſhall be ‡ 9. 11. alſo ‡ parallel to KB: But BO, KS, join them. * 7. 11. Therefore KBOS is * a quadrilateral Figure in one Plane: For if two Right Lines be parallel, and Points be taken in both of them, a Right Line joining the ſaid Points is in the ſame Plane as the Parallels are. And for the ſame Reaſon, each of the quadrilateral Figures SOPT, TPRY, are in one Plane. † 2. 11. And the Triangle YRX is † in one Plane. Therefore, if Right Lines be ſuppoſed to be drawn from the Points O, S, P, T, R, Y, to the Point A, there will be conſtituted a certain ſolid polyhedrous Figure within the Circumferences BX, KX, compoſed of Pyramids, whoſe Baſes are the quadrilateral Figures KBOS, SOPT, TPRY, and the Triangle YRX; and Vertices the Point A. And if there be made the ſame Conſtruction on each of the Sides KL, LM, ME, like as we have done on the Side KB, and alſo in the other three Quadrants, and the other Hemiſphere, there will be conſtituted a polyhedrous Figure deſcribed in the Sphere, compoſed of Pyramids whoſe Baſes are the aforeſaid quadrilateral Figures; and the Triangle YRX, being of the ſame Order and Vertices as the Point A; I ſay, the ſaid Polyhedron does not touch the Superficies of the Sphere, wherein the Circle FGH is. Let AZ be drawn‡ ‡ 11. 11. from the Point A, perpendicular to the Plane of the quadrilateral Figure KBSO, meeting it in the Point Z; and join BZ, ZK. Then, ſince AZ is perpendicular to the Plane of the quadrilateral Figure KBSO, it ſhall alſo be * perpendicular to all Right Lines that †*Def.* 3. 11. touch it, and are in the ſame Plane. Wherefore AZ is perpendicular to BZ and ZK. And becauſe AB is equal to AK, the Square of AB ſhall be alſo equal to the Square of AK: And the Squares of AZ, ZB, are † equal to the Square of AB. For the Angle at † 47. 1. Z is a Right Angle. And the Squares of AZ, ZK, are equal to the Square of AK. Therefore the Squares of AZ, ZB, are equal to the Squares of AZ, ZK. Let the common Square of AZ be taken away. And then the Square of BZ remaining is equal to the Square of ZK remaining: And ſo the Right Line BZ is equal to the Right Line ZK. After the ſame manner we demonſtrate, that Right Lines drawn from the Point Z to the Points O, S, are each equal to BZ, ZK. Therefore a Circle deſcribed about the Centre Z, with either of the Diſtances ZB, ZK, will alſo paſs thro' the Points O, S. And becauſe BKSO is a quadrilateral Figure in a Circle, and OB, BK, KS, are equal, and OS is leſs than BK; the Angle BZK ſhall be obtuſe; and ſo BK greater than BZ. But GL alſo is much greater than BK. Therefore GL is greater than BZ. And the Square of GL is greater than the Square of BZ. And ſince AL is equal to AB, the Square of AL ſhall be equal to the Square of AB. But the Squares of AG, GL, together, are equal to the Square of AL; and the Squares of BZ, ZA, together, equal to the Square of AB: Therefore the Squares of AG, GL, together, are equal to the Squares of BZ, ZA, together: But the Square of BZ is leſs than the Square of GL: Therefore the Square of ZA is greater than the Square of AG; and ſo the Right Line ZA will be greater than the Right Line AG. But AZ is perpendicular to one Baſe of the Polyhedron, and AG to the Superficies. Wherefore the Polyhedron does not touch the Superficies of the leſſer Sphere. Therefore

fore *there is deſcribed a ſolid Polyhedron in the greater, of two Spheres having the ſame Centre, which doth not touch the Superficies of the leſſer Sphere*; which was to be demonſtrated.

Coroll. Alſo if a ſolid Polyhedron be deſcribed in ſome other Sphere, ſimilar to that which is deſcribed in the Sphere BCDE; the ſolid Polyhedron deſcribed in the Sphere BCDE, to the ſolid Polyhedron deſcribed in that other Sphere, ſhall have a triplicate Proportion of that which the Diameter of the Sphere BCDE hath to the Diameter of that other Sphere. For the Solids being divided into Pyramids, equal in Number, and of the ſame Order, the ſame Pyramids ſhall be ſimilar. But ſimilar Pyramids are to each other in a triplicate Proportion of their homologous Sides. Therefore the Pyramid whoſe Baſe is the quadrilateral Figure KBOS, and Vertex the Point A, to the Pyramid of the ſame Order into the other Sphere, has a triplicate Proportion of that which the homologous Side of one has to the homologous Side of the other; that is, which AB, drawn from the Centre A of the Sphere, to that Line which is drawn from the Centre of the other Sphere. In like manner, every one of the Pyramids, that are in the Sphere whoſe Centre is A, to every one of the Pyramids of the ſame Order in the other Sphere, hath a triplicate Proportion of that which AB has to that Line drawn from the Centre of the other Sphere. And as one of the Antecedents is to one of the Conſequents, ſo are all the Antecedents to all the Conſequents. Wherefore the whole ſolid Polyhedron, which is in the Sphere deſcribed about the Centre A, to the whole ſolid Polyhedron that is in the other Sphere, hath a triplicate Proportion of that which AB hath to the Line drawn from the Centre of the other Sphere; that is, which the Diameter BD has to the Diameter of the other Sphere.

PROPOSITION XVIII.

THEOREM.

Spheres are to one another in a triplicate Proportion of their Diameters.

SUppoſe ABC, DEF, are two Spheres, whoſe Diameters are BC, EF. I ſay, the Sphere ABC to the Sphere DEF has a triplicate Proportion of that which BC has to EF.

For if it be not ſo, the Sphere ABC to a Sphere either leſſer or greater than DEF, will have a triplicate Proportion of that which BC has to EF. Firſt, let it be to a leſſer as GHK. And ſuppoſe the Sphere DEF to be deſcribed about the Sphere GHK; and let there be deſcribed * a ſolid Polyhedron in the greater Sphere DEF, not touching the Superficies of the leſſer Sphere GHK; alſo let a ſolid Polyhedron be deſcribed in the Sphere ABC, ſimilar to that which is deſcribed in the Sphere DEF. Then the ſolid Polyhedron in the Sphere ABC, to the ſolid Polyhedron in the Sphere DEF, will have † a triplicate Proportion of that which BC has to EF: But the Sphere ABC to the Sphere GHK, hath a triplicate Proportion of that which BC hath to EF. Therefore, as the Sphere ABC is to the Sphere GHK, ſo is the ſolid Polyhedron in the Sphere ABC to the ſolid Polyhedron in the Sphere DEF; and (by Inverſion) as the Sphere ABC is to the ſolid Polyhedron that is in it, ſo is the Sphere GHK to the ſolid Polyhedron that is in the Sphere DEF, but the Sphere ABC is greater than the ſolid Polyhedron that is in it. Therefore the Sphere GHK is alſo greater than the ſolid Polyhedron that is in the Sphere DEF, and alſo leſs than it, as being comprehended thereby, which is abſurd. Therefore the Sphere ABC to a Sphere leſs than the Sphere DEF, hath not a triplicate Proportion of that which BC has to EF. After the ſame manner it is demonſtrated that the Sphere DEF to a Sphere leſs than ABC, has not a triplicate Proportion of that which EF has to BC. I ſay, moreover, that the Sphere ABC to a Sphere greater than DEF, hath not a triplicate

* 17 *of this.*

† Cor. *to the laſt Prop.*

plicate Proportion of that which B C has to E F; for, if it be poſſible, let it have to the Sphere L M N greater than D E F. Then (by Inverſion) the Sphere L M N to the Sphere A B C, ſhall have a triplicate Proportion of that which the Diameter E F has to the Diameter B C; but as the Sphere L M N is to the Sphere A B C, ſo is the Sphere D E F to ſome Sphere leſs than A B C, becauſe the Sphere L M N is greater than D E F. Therefore the Sphere D E F to a Sphere leſs than A B C, hath a triplicate Proportion to that which E F has to B C, which is abſurd, and has been before proved. Therefore the Sphere A B C to a Sphere greater than D E F, has not a triplicate Proportion of that which B C has to E F. But it has alſo been demonſtrated, that the Sphere A B C to a Sphere leſs than D E F, has not a triplicate Proportion of that which B C has to E F. Therefore the Sphere A B C to the Sphere D E F, has a triplicate Proportion of that which B C has to E F; *which was to be demonſtrated.*

FINIS.

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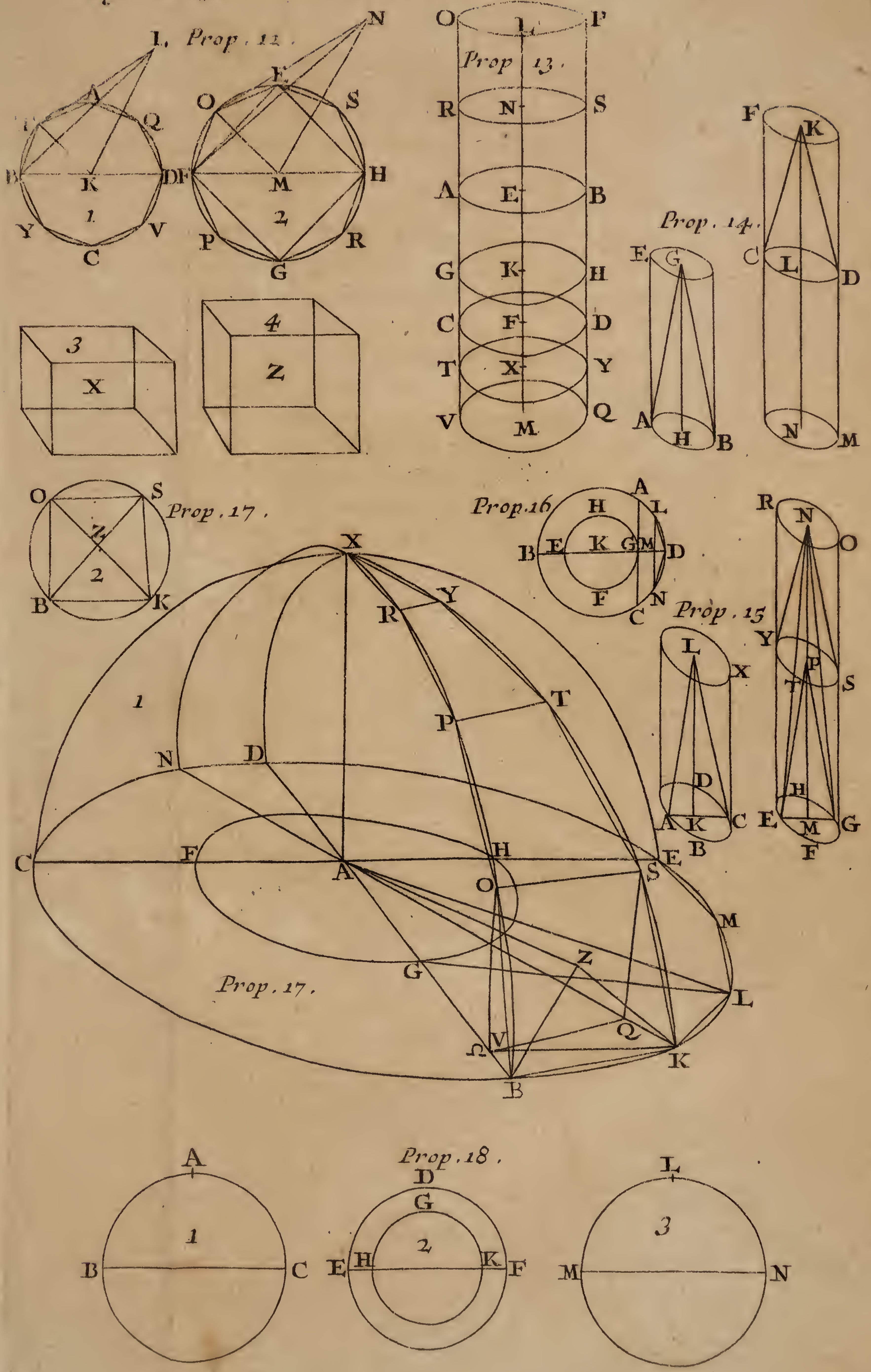


THE ELEMENTS Of *Plain* and *Spherical* TRIGONOMETRY.

DEFINITIONS.

THE Buſineſs of Trigonometry is to find the Angles when the Sides are given, and the Sides, or the Ratio's of the Sides, when the Angles are given, and to find Sides and Angles, when Sides and Angles are given: In order to which, it is neceſſary, that not only the Peripheries of Circles, but alſo certain Right Lines in and about Circles, be ſuppoſed divided into ſome determined Number of Parts.

And ſo the ancient Mathematicians thought fit to divide the Periphery of a Circle into 360 *Parts, which they call Degrees; and every Degree into* 60 *Minutes, and every Minute into* 60 *Seconds; and again, every Second into* 60 *Thirds, and ſo on. And every Angle is ſaid to be of ſuch a Number of Degrees and Minutes, as there are in the Arc meaſuring that Angle.*

There are ſome that would have a Degree divided into centeſimal Parts, rather than ſexageſimal ones: And it would perhaps be more uſeful to divide, not only a Degree, but even the whole Circle, in a decuple Ratio; which Diviſion may ſome time or other gain Place. Now, if a Circle contains 360 *Degrees, a Quadrant thereof, which is the Meaſure of a Right Angle, will* *be* 90 *of thoſe Parts: And if it contains* 100 *Parts, a Quadrant will be* 25 *of theſe Parts.*

The Complement *of an Arc is the Difference thereof from a Quadrant.*

A Chord, *or* Subtenſe, *is a Right Line drawn from one End of the Arc to the other.*

The Right Sine *of any Arc, which alſo is commonly called only a Sine, is a Perpendicular falling from one End of an Arc, to the Radius drawn through the other End of the ſaid Arc. And is therefore the Semiſubtenſe of double the Arc,* viz. $DE=\frac{1}{2}DO$, *and the Arc* DO *is double of the Arc* DB. *Hence, the Sine of an Arc of* 30 *Degrees is equal to the one half of the Radius. For* (by 15. *El.* 4.) *the Side of an Hexagon inſcrib'd in a Circle, that is, the Subtenſe of* 60 *Degrees, is equal to the Radius. A Sine divides the Radius into two Segments* CE, EB; *one of which,* CE, *which is intercepted between the Centre and the Right Sine, is the Sine Complement of the Arc* DB *to a Quadrant, (for* $CE=FD$, *which is the Sine of the Arc* DH) *and is called the* Coſine. *The other Segment* BE, *which is intercepted between the Right Sine and the Periphery, is called a* verſed Sine, *and ſometimes a* Sagitta.

And if the Right Line CG *be produced from the Centre C, thro' one End* D *of the Arc, until it meets the Right Line* BG, *which is perpendicular to the Diameter drawn thro' the other End* B *of the Arc, then* CG *is called the* Secant, *and* BG *the* Tangent *of the Arc* DB.

The Coſecant and Cotangent of an Arc is the Secant or Tangent of that Arc, which is the Complement of the former Arc to a Quadrant. Note, *As the Chord of an Arc, and of its Complement to a Circle, is the ſame; ſo likewiſe is the Sine, Tangent, and Secant of an Arc, the ſame as the Sine, Tangent, and Secant of its Complement to a Semicircle.*

The Sinus Totus *is the greateſt Sine, or the Sine of* 90 *Degrees, which is equal to the Radius of the Circle.*

A Trigonometrical Canon *is a Table, which, beginning from one Minute, orderly expreſſes the Lengths that every Sine, Tangent, and Secant, have in reſpect of the Radius, which is ſuppoſed Unity; and is conceived to be divided* 10,000,000, *or more decimal Parts. And* *ſo the Sine, Tangent, or Secant of any Arc, may be had by Help of this Table; and contrarywiſe, a Sine, Tangent, or Secant, being given, we may find the Arc it expreſſes.* Take Notice, *That in the following Tract,* R. *ſignifies the Radius,* S. *a Sine,* Coſ. *a Coſine,* T. *a Tangent, and* Cot. *a Cotangent.*

The CONSTRUCTION of the *Trigonometrical Canon.*

PROPOSITION I.

THEOREM.

The two Sides of any Right-angled Triangle being given, the other Side is alſo given.

FOR (by 47. of the firſt *Element*) $ACq = ABq + BCq$ and $ACq - BCq = ABq$, and interchangeably $ACq - ABq = BCq$. Whence, by the Extraction of the ſquare Root, there is given $AC = \sqrt{ABq + BCq}$ and $AB = \sqrt{ACq - BCq}$. And $BC = \sqrt{ACq - ABq}$.

PROPOSITION II.

PROBLEM.

The Sine DE *of the Arc* BD, *and the Radius* CD, *being given, to find the Coſine* DF.

THE Radius CD, and the Sine DE, being given in the Right-angled Triangle CDE, there will be given (by the laſt *Prop.*) $\sqrt{CDq - DEq} = DF$.

PROPOSITION III.

PROBLEM.

The Sine DE *of any Arc* DB *being given, to find* DM *or* BM *the Sine of half the Arc.*

DE being given, CE (by the laſt *Prop.*) will be given, and accordingly EB, which is the Difference between the Coſine and Radius. Therefore DE, EB, being given in the Right-angled Triangle DBE, there will be given DB, whoſe Half DM is the Sine of the Arc DL $= \frac{1}{2}$ the Arc BD.

PROPOSITION IV.

PROBLEM.

The Sine BM *of the Arc* BL *being given, to find the Sine of double that Arc.*

THE Sine BM being given, there will be given (by *Prop.* 2.) the Coſine CM. But the Triangles CBM, DBE, are equiangular, becauſe the Angles at E and M are Right Angles, and the Angle at B common. Wherefore (by 4. 6.) we have CB : CM : : BD, or 2 BM : DE. Whence, ſince the three firſt Terms of this Analogy are given, the fourth alſo, which is the Sine of the Arc DB, will be known.

Coroll. Hence, CB : 2 CM : : BD : 2 DE; that is, the Radius is to double the Coſine of one Half of the Arc DB, as the Subtenſe of the Arc DB is to the Subtenſe of double that Arc. Alſo CB : 2 CM : : (2 BM : 2 DE : :) BM : DE : : $\frac{1}{2}$ CB : CM. Wherefore the Sine of any Arc, and the Sine of its Double, being given, the Coſine of the Arc itſelf is given.

PROPOSITION V.

The Sines of two Arcs BD, FD, *being given, to find* FI *the Sine of the Sum, as likewiſe* EL, *the Sine of their Difference.*

LET the Radius CD be drawn, and then CO is the Coſine of the Arc FD, which accordingly is given, and draw OP thro' O parallel to DK. Alſo let OM, GE, be drawn parallel to CB. Then, becauſe the Triangles CDK, COP, CHI, FOH, FOM, are equiangular. In the firſt Place, CD : DK : : CO : OP, which conſequently is known. Alſo we have CD : CK : : FO : FM, and ſo likewiſe this ſhall be known. But becauſe FO = EO, then will FM = MG = ON. And ſo OP + FM = FI = Sine of the Sum of the Arcs: And OP — FM; that is, OP — ON = EL = Sine of the Difference of the Arcs. W. W. D.

Coroll. Becauſe the Differences of the Arcs BE, BD, BF, are equal, the Arc BD ſhall be an Arithmetical Mean between the Arcs BE, BF.

PROPOSITION VI.

The ſame Things being ſuppoſed, Radius is to double the Coſine of the mean Arc, as the Sine of the Difference to the Difference of the Sines of the Extremes.

FOR we have CD : CK : : FO : FM; whence by doubling the Conſequents CD : 2 CK : : FO : 2 FM, or to FG; which is the Difference of the Sines EL, FI. W. W. D.

Coroll. If the Arc BD be 60 Degrees, the Difference of the Sines FI, EL, ſhall be equal to the Sine FO, of the Diſtance. For in this Caſe, CK is the Sine of 30 Degrees; double thereof being equal to Radius: And ſo, ſince CD = 2 CK, we ſhall have FO = FG. And conſequently, if the two Arcs BE, BF, are equidiſtant from the Arc of 60

Degrees, the Difference of the Sines ſhall be equal to the Sine of the Diſtance F D.

Coroll. 2. Hence, if the Sines of all Arcs be given diſtant from one another by a given Interval, from the Beginning of a Quadrant to 60 Degrees, the other Sines may be found by one Addition only. For the Sine of 61 Degrees = Sine of 59 Degrees + Sine of 1 Degree. And the Sine of 62 Degrees = Sine of 58 Degrees + Sine of 2 Degrees. Alſo the Sine of 63 Degrees = Sine of 57 Degrees + Sine of 3 Degrees, and ſo on.

Coroll. 3. If the Sines of all Arcs, from the Beginning of a Quadrant to any Part of the Quadrant, diſtant from each other, by a given Interval, be given, thence we may find the Sines of all Arcs to the Double of that Part. For Example, Let all the Sines to 15 Degrees be given; then, by the precedent Analogy, all the Sines to 30 Degrees, may be found. For Radius is to double the Coſine of 15 Degrees, as the Sine of 1 Degree is to the Difference of the Sines of 14 Degrees, and 16 Degrees; ſo alſo is the Sine of 3 Degrees, to the Difference between the Sines of 12 and 18 Degrees; and ſo on continually, until you come to the Sine of 30 Degrees.

After the ſame manner, as Radius is to double the Coſine of 30 Degrees, or to double the Sine of 60 Degrees, ſo is the Sine of 1 Degree to the Difference of the Sines of 29 and 31 Degrees : : Sine 2 Degrees, to the Difference of the Sines of 28 and 32 Degrees : : Sine 3 Degrees, to the Difference of the Sines of 27 and 33 Degrees. But in this Caſe, Radius is to double the Coſine of 30 Degrees, as 1 to $\sqrt{3}$ *. And accordingly, if the

FIG. *for the* DEFINITIONS.

* Let B D be an Arch of 30^{es}

Rad. Tan. Co-ſine Sine

Then as CB : BG : : FD : DE. DO = CB *ergo* DE = $\frac{1}{2}$.

$\sqrt{CBq - DEq} = CE = \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}}$. CB : CE : 1 : $\sqrt{\frac{3}{4}}$.

CD : 2 CE : : 1 : $2\sqrt{\frac{3}{4}} = \sqrt{\frac{12}{4}}\sqrt{3}$. *Q. E. D.*

Sines of the Diſtances from the Arc of 30 Degrees, be multiplied by $\sqrt{3}$, the Differences of the Sines will be had.

So likewiſe may the Sines of the Minutes in the Beginning of the Quadrant be found, by having the Sines and Coſines of one and two Minutes given. For, as the Radius is to double the Coſine of 2 : : Sine 1 : Difference of the Sines of 1′ and 3′ : : Sine 2′ : Difference of the Sines of 0′ and 4′, that is, to the Sine of 4′. And ſo the Sines of the four firſt Minutes being given, we may thereby find the Sines of the others to 8′, and from thence to 16′, and ſo on.

PROPOSITION VII.

THEOREM.

In ſmall Arcs, the Sines and Tangents of the ſame Arcs are nearly to one another, in a Ratio of Equality.

FOR, becauſe the Triangles CED, CBG, are equiangular, CE : CB : : ED : : BG. But as the Point E approaches B, EB will vaniſh in reſpect of the Arc BD: Whence CE will become nearly equal to CB. And ſo ED will be alſo nearly equal to BG. If EB be leſs than the $\frac{1}{10,000,000}$ Part of the Radius, then the Difference between the Sine and the Tangent will be alſo leſs than the $\frac{1}{10,000,000}$ Part of the Tangent.

Coroll. Since any Arc is leſs than the Tangent, and greater than its Sine, and the Sine and Tangent of a very ſmall Arc are nearly equal; it follows that the Arc ſhall be nearly equal to its Sine; and ſo in very ſmall Arcs it ſhall be, as Arc is to Arc, ſo is Sine to Sine.

PROPOSITION VIII.

To find the Sine of the Arc of one Minute.

THE Side of a Hexagon inſcrib'd in a Circle, that is, the Subtenſe of 60 Degrees, is equal to the Radius, (*by* 15*th* of the 4*th*) and ſo the Half of the Radius ſhall be the Sine of the Arc 30 Degrees. Wherefore the Sine of the Arc of 30 Degrees being given, the Sine of the Arc of 15 Degrees may be found (*by Prop.* 3.). Alſo the Sine of the Arc of 15 Degrees being given, (*by the ſame Prop.*) we may have the Sine of 7 Degrees 30 Minutes: So likewiſe can we find the Sine of the Half of this, *viz.* 3 Degrees $45'$; and ſo on, until twelve Biſections being made, we come to an Arc of 52^{2}, 44^{3}, 03^{4}, 45^{5}, whoſe Coſine is nearly equal to the Radius, in which Caſe (*as is manifeſt from Prop.* 7.) Arcs are proportional to their Sines: And ſo, as the Arc of 52^{2}, 44^{3}, 3^{4}, 45^{5}, is to an Arc of one Minute, ſo ſhall the Sine before found, be to the Sine of an Arc of one Minute, which therefore will be given. And when the Sine of one Minute is found, then (*by Prop.* 2. *and* 4.) the Sine and Coſine of two Minutes will be had.

PROPOSITION IX.

THEOREM.

If the Angle BAC, *being in the Periphery of a Circle, be biſected by the Right Line* AD, *and if* AC *be produced until* DE = AD *meets it in* E: *then ſhall* CE = AB.

IN the quadrilateral Figure ABDC (*by* 22, 3.) the Angles B and ACD are equal to two Right Angles = DCE + DCA (*by* 13. 1.) Whence the Angle B = DCE. But likewiſe the Angle E = DAC (*by* 5. 1.) = DAB, and DC = DB. Wherefore the Triangles BAD and CED are congruous, and CE is equal to AB. W. W. D.

PROPOSITION X.

THEOREM.

Let the Arcs AB, BC, CD, DE, EF, &c. *be equal; and let the Subtenſes of the Arcs* AB, AC, AD, AE, &c. *be drawn; then will* AB : AC : : AC : AB + AD : : AD : AC + AE : : AE : AD + AF : : AF : AE + AG.

LET AD be produced to H, AE to I, AF to K, and AG to L, that the Triangles ACH, ADI, AEK, AFL, be Iſoſceles ones; then, becauſe the Angle BAD is biſected, we ſhall have DH = AB (*by the laſt Prop.*); ſo likewiſe ſhall EI = AC, FK = AD, alſo GL = AE.

But the Iſoſceles Triangles ABC, CHA, DAI, EAK, FAL, becauſe of the equal Angles at the Baſes, are equiangular. Wherefore it ſhall be as AB : AC : : AC : AH = AB + AD : : AD : AI = AC + AE : : AE : AK = AD + AF : : AF : AL = AE + AG. W. W. D.

Coroll. 1. Becauſe AB is to AC, as Radius is to double the Coſine of $\frac{1}{2}$ the Arc AB, it ſhall alſo be (by *Coroll. Prop.* 4.) as Radius is to double the Coſine of $\frac{1}{2}$ the Arc AB, ſo is $\frac{1}{2}$ AB : $\frac{1}{2}$ AC : : $\frac{1}{2}$ AC : : $\frac{1}{2}$ AB + $\frac{1}{2}$ AD : : $\frac{1}{2}$ AD : $\frac{1}{2}$ AC + $\frac{1}{2}$ AE : : $\frac{1}{2}$ AE + $\frac{1}{2}$ AD + $\frac{1}{2}$ AF, *&c.* Now let each of the Arcs AB, BC, CD, *&c.* be $2'$; then will $\frac{1}{2}$ AB be the Sine of one Minute, $\frac{1}{2}$ AC the Sine of $2'$ Minutes, $\frac{1}{2}$ AD the Sine of $3'$ Minutes; $\frac{1}{2}$ AE the Sine of $4'$, *&c.* Whence, if the Sines of one and two Minutes be given, we may eaſily find all the other Sines in the following manner.

Let the Coſine of the Arc of one Minute, that is, the Sine of the Arc of 89 Deg. $59'$, be called Q, and make the following Analogies; R : 2 Q : : Sin. $2'$: S. $1'$ + S. $3'$. Wherefore the Sine of 3 Minutes will be given. Alſo R : 2 Q : : S. $3'$: S. $2'$ + S. $4'$. Wherefore the S. $4'$ is given; and R. 2 Q : : S. $4'$: S. $3'$ + S. $5'$; and ſo the Sine of $5'$ will be had.

Likewiſe $R : 2\,Q :: S.\,5' : S.\,4' + S.\,6'$; and ſo we ſhall have the Sine of 6′. And in like manner, the Sines of every Minute of the Quadrant will be given. And becauſe the Radius, or the firſt Term of the Analogy, is Unity, the Operations will be with great Eaſe and Expedition calculated by Multiplication, and contracted by Addition. When the Sines are found to 60 Degrees, all the other Sines may be had by Addition only (by *Cor.* 1. *Prop.* 6.).

The Sines being given, the Tangents and Secants may be found from the following Analogies (in the Figure for the Definitions); becauſe the Triangles CED, CBG, CHI, are equiangular, we have

CE : ED :: CB : BG; that is, Coſ. : S :: R : T
GB : BC :: CH : HI; that is, T : R :: R : Cot.
CE : CD :: CB : CG; that is, Coſ. : R :: R : Secant
DE : CD :: CH : CI; that is, S : R :: R : Coſec.

SCHOLIUM.

That great Geometrician and incomparable Philoſopher, Sir Iſaac Newton, *was the firſt that laid down a Series converging,* in infinitum; *from which, having the Arcs given, their Sines may be found. Thus, if an Arc be called* A, *and the Radius be an Unit, the Sine thereof will be found to be*

$$A - \frac{A^3}{1.2.3} + \frac{A^5}{1.2.3.4.5} - \frac{A^7}{1.2.3.4.5.6.7} + \frac{A^9}{1.2.3.4.5.6.7.8.9}\ \&c.$$

And the Coſine,

$$1 - \frac{A^2}{1.2.3} + \frac{A^4}{1.2.3.4} - \frac{A^6}{1.2.3.4.5.6} + \frac{A^8}{1.2.3.4.5.6.7.8}\ \&c.$$

Theſe Series in the Beginning of the Quadrant, when the Arc A *is but ſmall, ſoon converge. For in the Series for the Sine, if* A *does not exceed* 10 *Minutes, the two firſt Terms thereof,* viz. $A - \frac{1}{6}A^3$ *gives the Sine to* 15 *Places of Figures. If the Arc* A *be not greater than one Degree, the three firſt Terms will exhibit the Sine to* 15 *Places of Figures; and ſo the ſaid Series are very uſeful for finding the firſt and laſt Sines of the Quadrant. But the greater the Arc* A *is, the more* *are the Terms of the Series required to have the Sine in Numbers true to a given Place of Figures. And then, when the Arc is nearly equal to the Radius, the Series converges very ſlow. And therefore, to remedy this, I have deviſed other Series, ſimilar to the* Newtonian *one, wherein, I ſuppoſe, the Arc, whoſe Sine is ſought, is the Sum and Difference of two Arcs,* viz. $A+z$, *or* $A-z$: *And let the Sine of the Arc* A *be called* a, *and the Coſine* b. *Then the Sine of the Arc* $A+z$ *will be expreſſed thus:*

$$1.\ a+\frac{bz}{1}-\frac{az^2}{1.2}-\frac{bz^3}{1.2.3}+\frac{az^4}{1.2.3.4}+\frac{bz^5}{1.2.3.4.5}\ \&c.$$

And the Coſine is,

$$2.\ b-\frac{az}{1}-\frac{bz^2}{1.2}+\frac{az^3}{1.2.3}+\frac{bz^4}{1.2.3.4}-\frac{az^5}{1.2.3.4.5}-\frac{bz^6}{1.2.3.4.5.6}\ \&c.$$

In like manner the Sine of the Arc $A-z$ *is,*

$$3.\ a-\frac{bz}{1}-\frac{az^2}{1.2}+\frac{az^3}{1.2.3}+\frac{az4}{1.2.3.4}-\frac{bz^5}{1.2.3.4.5}-\frac{az^6}{1.2.3.4.5.6}\ \&c$$

And the Coſine is,

$$4.\ b+\frac{az}{1}-\frac{bz^2}{1.2}-\frac{az^3}{1.2.3}+\frac{bz^4}{1.2.3.4}+\frac{az^5}{1.2.3.4.5}\ \&c.$$

The Arc A *is an Arithmetical Mean between the Arc* $A-z$ *and* $A+z$. *And the Difference of the Sines are,*

$$5.\ \frac{bz}{1}-\frac{az^2}{1.2}-\frac{bz^3}{1.2.3}+\frac{az^4}{1.2.3.4}+\frac{bz^5}{1.2.3.4.5}-\frac{az^6}{1.2.3.4.5.6}\ \&c.$$

$$6.\ \frac{bz}{1}+\frac{az^2}{1.2}-\frac{bz^3}{1.2.3}-\frac{az^4}{1.2.3.4}+\frac{bz^5}{1.2.3.4.5}+\frac{az^6}{1.2.3.4.5.6}\ \&c.$$

Whence the Difference of the Differences or ſecond Difference.

$$7.\ \textit{Produce}\ \underline{\frac{2az^2}{1.2}-\frac{2az^4}{1.2.3.4}+\frac{2az^6}{1.2.3.4.5.6}}\ \&c.$$

$$\text{Or } 2a\times\frac{z^2}{1.2}-\frac{z^4}{1.2.3.4}+\frac{z^6}{1.2.3.4.5.6}\ \&c.$$

Which Series is equal to double the Sine of the Mean Arc, drawn into the verſed Sine of the Arc z, *and converges very ſoon. So that if* z *be the firſt Minute of the Quadrant, the firſt Term of the Series gives the ſecond Difference to* 15 *Places of Figures, and the ſecond Term to* 25 *Places.*

From hence, if the Sines of the Arcs diſtant one Minute from each other be given, the Sines of all the Arcs that are in the ſame Progreſſion may be found by an exceeding eaſy Operation.

In the firſt and ſecond Series, if A=0; *then ſhall* a=0, *and* b *its Coſine, will become Radius, or* 1. *And hence, if the Terms wherein* a *is, are taken away, and* 1 *to be put inſtead of* b, *the Series will become the* Newtonian. *In the third and fourth Series, if* A *be* 90 *Degrees, we ſhall have* b=0, *and* a=1. *Whence again, taking away all the Terms wherein* b *is, and putting* 1 *inſtead of* a, *we ſhall have the* Newtonian *Series ariſe.*

Note, *All the ſaid Series eaſily flow from the* Newtonian *ones. By the fifth* Propoſition.

PROPOSITION XI.

In a Right-angled Triangle, if the Hypothenuſe be made the Radius, then are the Sides the Sines of their oppoſite Angles; and if either of the Legs be made the Radius, then the other Leg is the Tangent of its oppoſite Angle, and the Hypothenuſe is the Secant of that Angle.

IT is manifeſt, that CB is the Sine of the Arc CD, and AB the Coſine thereof; but the Arc CD is the Meaſure of the Angle A, and the Complement of the Meaſure of the Angle C. Moreover, if AB, in the Figure to this Propoſition, be ſuppoſed Radius, then BC is the Tangent, and AC the Secant of the Arc BD, which is the Meaſure of the Angle A. So alſo, if BC be made the Radius, then is BA the Tangent, and AC the Secant of the Arc BE, or Angle C. W. W. D. Therefore, as AC being taken as ſome given Meaſure, is to BC taken in the ſame Meaſure; ſo ſhall the Number 10,000,000 Parts, in which the Radius is ſuppoſed to be divided, be to a Number expreſſing in the ſame Parts the Length of the Sine of the Angle A; that is, it will be

as AC : BC : R : S, A.
by the ſame Reaſon, as AC : BA : R : S, C.
alſo as AB : BC : R : T, A.
and BC : BA : R : T, C.

And ſo, if any three of theſe Proportionals be given, the fourth may be found by the *Rule of Three.*

PROPOSITION XII.

The Sides of plain Triangles are as the Sines of their oppoſite Angles.

IF the Sides of a Triangle, inſcribed in a Circle, be biſected by perpendicular Radii, then ſhall the half Sides be the Sines of the Angles at the Periphery; for the Angle BDC at the Centre, is double of the Angle BAC at the Periphery; (by 20 *El. lib.* 3.) and ſo the Half of every of them, *viz.* BDE = BAC, and BE is the Sine thereof. For the ſame Reaſon, BF ſhall be the Sine of the Angle BCA, and AG the Sine of the Angle ABC.

In a Right-angled Triangle we have $BD = \frac{1}{2} BC$ = Radius (*by* 31. *Eucl.* 3.) but Radius is the Sine of a Right Angle: Whence $\frac{1}{2}$ BC is the Sine of the Angle A.

In an Obtuſe-angled Triangle, let BI, CI, be drawn, and then the Angle L ſhall be the Complement of the Angle A to two Right Angles, (by 22 *El.* 3.) and ſo they ſhall both have the ſame Sine. But the Angle BDE, (whoſe Sine is BE) = Angle L. Therefore BE ſhall be the Sine of the Angle BAC. And ſo in every Triangle, the Halves of the Sides are the Sines of the oppoſite Angles; but it is manifeſt, that the Sides are to one another as their Halves. W. W. D.

PROPOSITION XIII.

In a plain Triangle, the Sum of the Legs, the Difference of the Legs, the Tangent of the half Sum of the Angles at the Baſe, and the Tangent of one half their Difference, are proportional.

LET there be a Triangle ABC, whoſe Legs are AB, BC, and Baſe AC. Produce AB to H, ſo that $BH = BC$; then ſhall AH be the Sum of the Legs; and if you make $BI = BA$, then IH will be the Difference of the Legs. Alſo the Angle $HBC = \text{Angles } A + ACB$, (*by* 32. *El.* 1.) and ſo EBC the Half thereof = half the Sum of the Angles A and ACB, and its Tangent (putting the Radius = EB) is EC. Again, let BD be drawn parallel to AC, and make $HF = CD$. Then, ſince $HB = CB$, we ſhall have (*by* 4. *El.* 1.) the Angle $HBF = CBD = BCA$. (*by* 29. *El.* 1.) Alſo the Angle $HBD = \text{Angle } A$; whence FBD ſhall be the Difference of the Angles A and ACB; and EBD, whoſe Tangent is ED, half their Difference. Let IG be drawn thro' I parallel to AC or BD, and then (*by* 2. *El.* 6.) $AB : BI :: CD : DG$: But $AB = BI$; whence we ſhall have $CD = DG$: But $CD = HF$, and ſo $HF = DG$, and conſequently, $HG = DF$, and $\frac{1}{2} HG = \frac{1}{2} DF = DE$; and becauſe the Triangles AHC, IHG, are equiangular, it ſhall be as $AH : IH :: HC : HG :: \frac{1}{2} HC : \frac{1}{2} HG :: EC : ED$. That is, AH the Sum of the Legs, to IH the Difference of the Legs, ſhall be as EC the Tangent of one half the Sum of the Angles at the Baſe, to ED the Tangent of one half their Difference. W. W. D.

PROPOSITION XIV.

In a plain Triangle, the Baſe, the Sum of the Sides, the Difference of the Sides, and the Difference of the Segments of the Baſe, are proportional.

LET DC be the Baſe of the Triangle BCD. About the Centre B, with the Radius BC, let a Circle be deſcribed. Produce DB to G, and from B let fall BE perpendicular to the Baſe; then ſhall $DG = DB + BC =$ Sum of the Sides, and $DH =$ Difference of the Sides; and DE, CE, are the Segments of the Baſe whoſe Difference is DF; becauſe (*by Cor. Prop.* 38. *El.* 3.) the Rectangle under DC and DF, is equal to the Rectangle under DG, DH, it ſhall be (*by* 16. *El.* 6.) as $DC : DG :: DH : DF$.

PROBLEM.

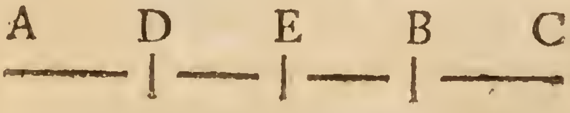
The Sum and Difference of any two Quantities being given, to find the Quantities themſelves.

IF one Half of the Sum be added to one Half of the Difference, the Aggregate ſhall be equal to the greater of the Quantities; and if from one Half of the Sum be taken one Half of the Difference, the Reſidue ſhall be equal to the leſſer of the Quantities. For, let there be two Quantities AB, BC, and let there be taken $AD = BC$; then DB will be their Difference, and AC their Sum; which, biſected in E, gives AE or EC the half Sum, and DE or EB the half Difference. Hence $AB = AE + EB =$ the half Sum + the half Difference, and $BC = CE - EB =$ the half Sum — the half Difference.

In any plain Triangle, if two Angles be given, the third Angle is alſo given, becauſe it is their Complement to two Right Angles.

If one of the acute Angles of a Right-angled Triangle be given, the other acute Angle will be given, becauſe it is the Complement of the given Angle to a Right Angle.

And if two Sides of a Right-angled Triangle be given, the other Side may be found by the firſt Propoſition without a Canon.



The Trigonometrical Solutions of a Right-angled Triangle, may be as follow. Vid. Fig. A.

	Given	Sought	Make as
1	The Legs AB and BC.	The Angles.	AB : BC :: R : T of the Angle A; whoſe Complement is the Angle C.
2	The Leg AE, and the Hypothenuſe AC.	The Angles.	AC : AB :: R : S, C, whoſe Complement is the Angle A.
3	The Leg AB, and the Angle A.	The other Side BC, and the Hypothenuſe AC.	R : T, A :: AB : BC. S, C : R :: AB : AC.
4	The Hypothenuſe AC, and the Angle C.	The Leg AB.	R :: S, C : AC : AB.

The Trigonometrical Solutions of Oblique-angled Triangles. Vid. Fig. to Prop. 12.

	Given	Sought	Make as
1	The Angles A, B, C, and the Side AB.	The Sides BC and AC.	S, C : S A :: AB : BC. Alſo, S, C : S, B :: AB : AC : But when two Angles are given, the third is alſo given; whence the Caſe wherein two Angles and a Side are given, to find the reſt, falls into this Caſe.

	Given	Sought	Make as
2	All the Angles A, B, C.	All the Sides AB, AC, BC.	S,C : S,A : : AB : BC. And S,C : S,B : : AB : AC : Whence, if the Angles are given, the Proportions of the Sides may be found, but not the Sides themſelves, unleſs one of them be firſt known.
3	The two Sides AB, BC, and C, the Angle oppoſite to one of them.	The Angles A and B.	AB : BC : : S,C : S,A; which therefore may be found. When AB the Side oppoſite to C, the given Angle, is longer than CB the Side oppoſite to the ſought Angle, the ſought Angle is leſs than a right one. But when it is ſhorter, becauſe the Sine of an Angle, and that of its Complement to two Right Angles, is the ſame, the Species of the Angle A muſt be firſt known, or the Solution will be ambiguous.
4	The two Sides AB, BC, and the interjacent Angle B.	The Angles A and C.	*Vid. Fig.* to *Prop.* 13. $BC+AB : BC-AB :: T,\frac{A+C}{2}\ T,\frac{A-C}{2}$ Whence is known the Difference of the Angles A and C, whoſe Sum is given; and ſo (by the *Prob.* following *Prop.* 14.) the Angles themſelves will be given.
5	All the Sides AB, BC, AC	The Angles.	*Vid. Fig.* B. Let the Perpendicular be drawn from the Vertex to the Baſe, and find the Segments of the Baſe by *Prop.* 14. *viz.* Make as BC : AC+AB : : AC—AB : DC—DB. And ſo BD, DC, are given from this Analogy; and thence the Angles ABD, ADC, will be given by the Reſolution of Right-angled Triangles.

THE ELEMENTS OF Spherical Trigonometry.

DEFINITIONS.

I. THE *Poles of a Sphere are two Points in the Superficies of the Sphere, that are the Extremes of the Axis.*

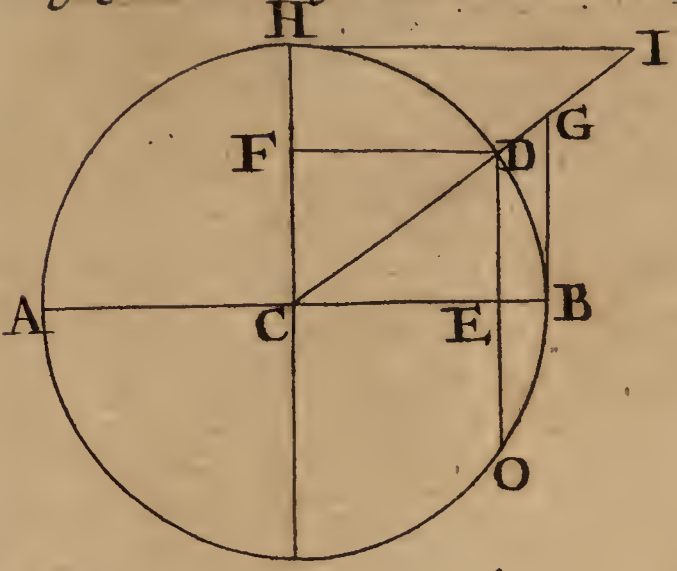
II. *The Pole of a Circle in a Sphere is a Point in the Superficies of the Sphere, from which all Right Lines that are drawn to the Circumference of the Circle, are equal to one another.*

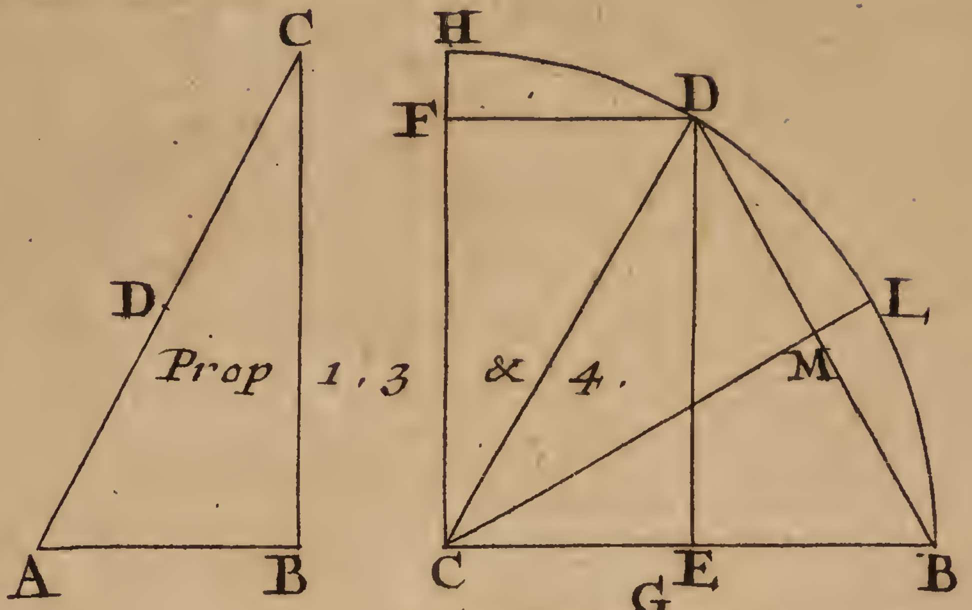
III. *A great Circle in a Sphere, is that whoſe Plane paſſes thro' the Centre of the Sphere, and whoſe Centre is the ſame of that of the Sphere.*

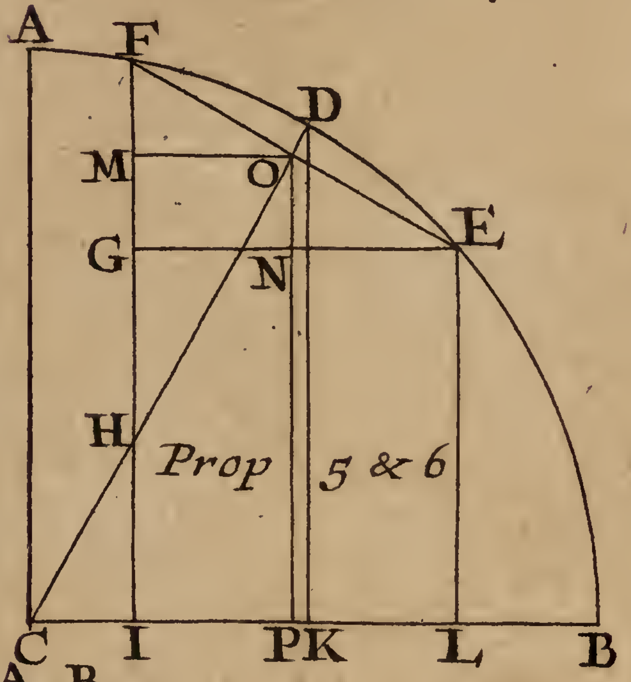
IV. *A ſpherical Triangle is a Figure comprehended under the Arcs of three great Circles in a Sphere.*

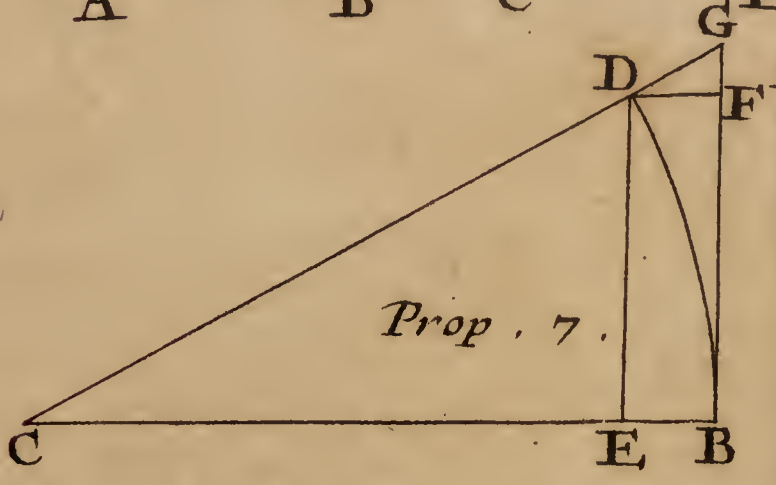
V. *A ſpherical Angle is that which, in the Superficies of the Sphere, is contained under two Arcs of great Circles; and this Angle is equal to the Inclination of the Planes of the ſaid Circles.*

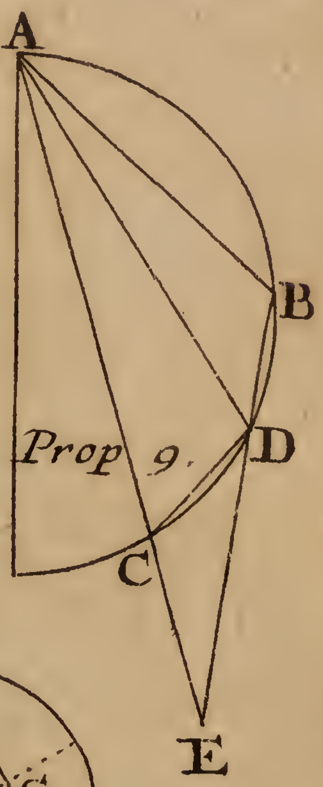
Fig. for the Definitions and Prop. 2.

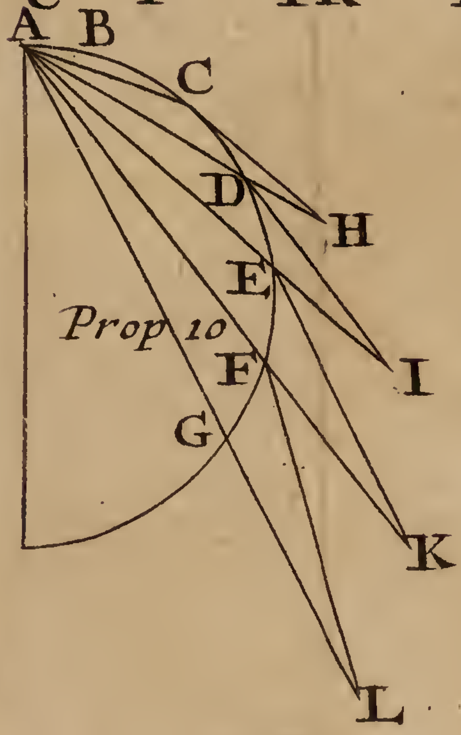


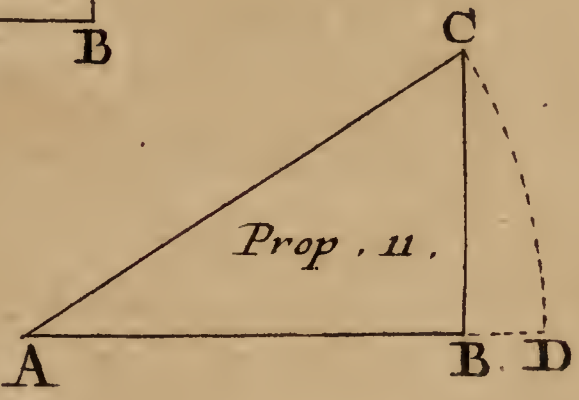


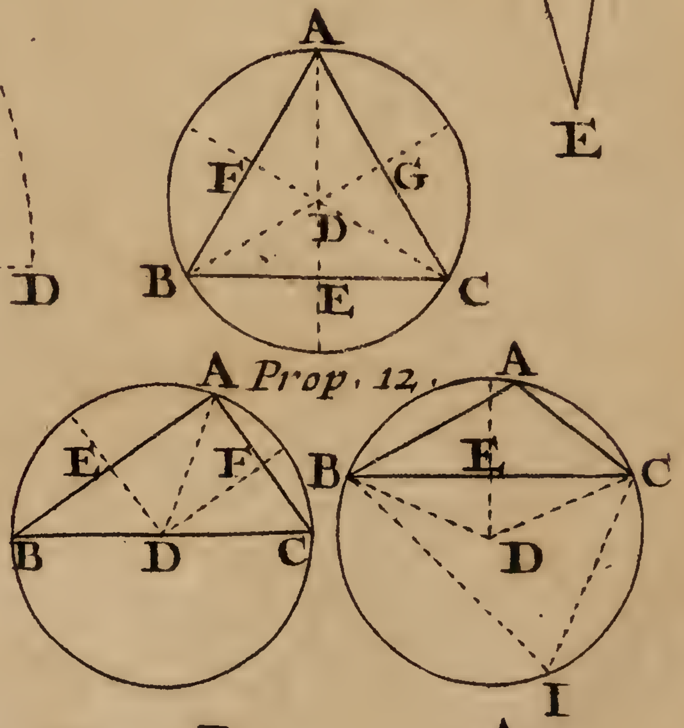


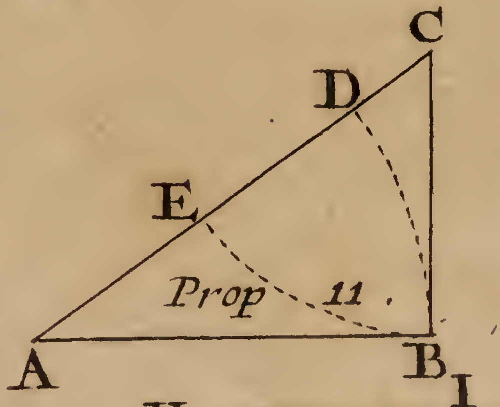


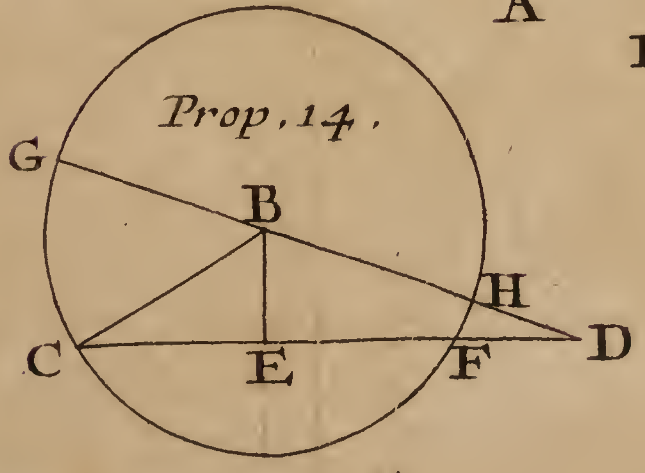


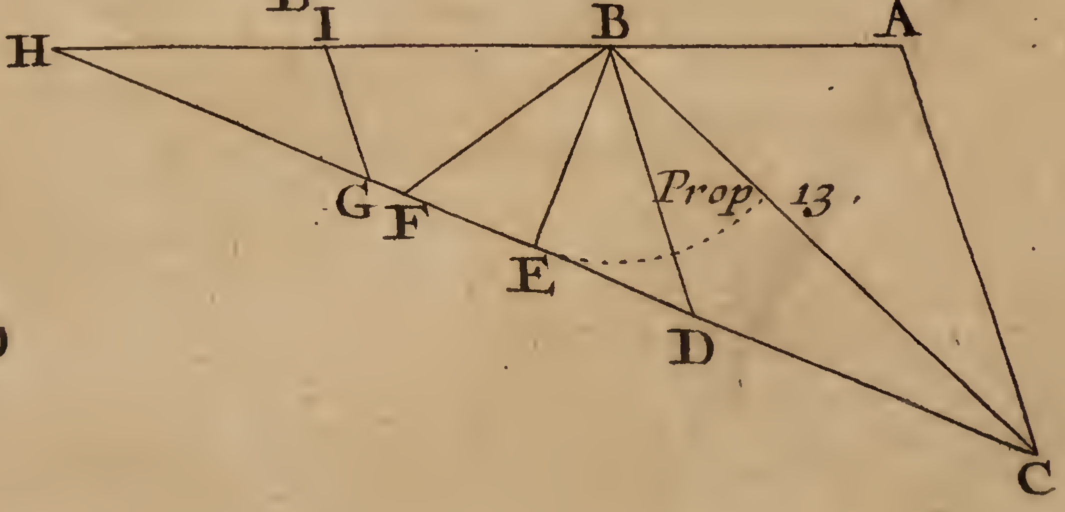


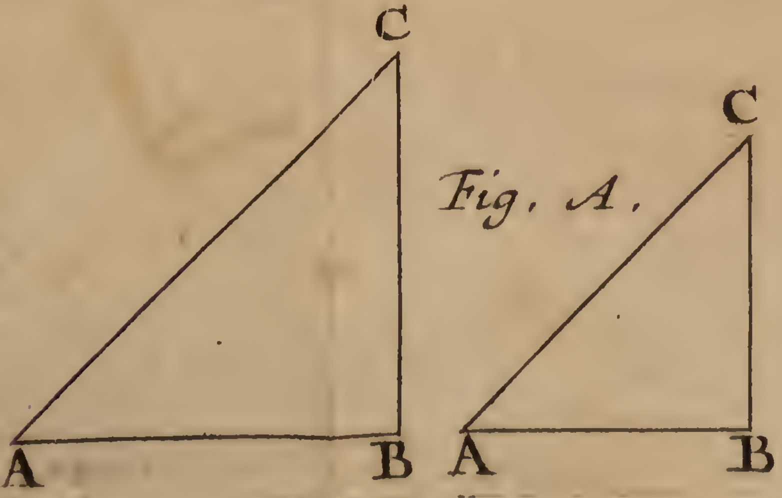


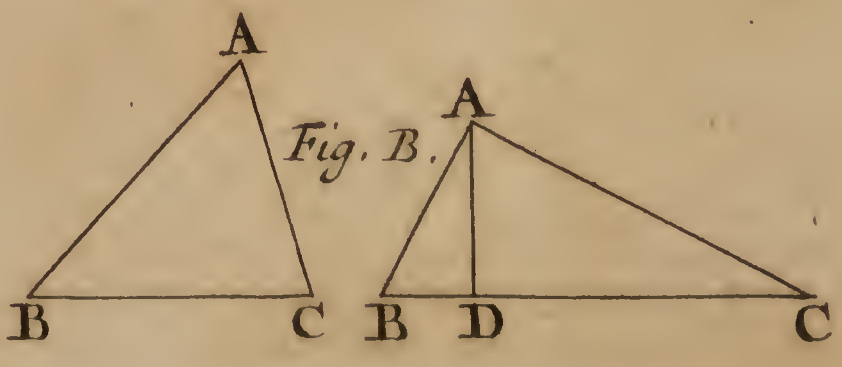












PROPOSITION I.

Great Circles ACB, AFB, *mutually biſect each other.*

FOR ſince the Circles have the ſame Centre, their common Section ſhall be a Diameter of each Circle, and ſo will cut them into two equal Parts.

Coroll. Hence the Arcs of two great Circles in the Superficies of the Sphere, being leſs than Semicircles, do not comprehend a Space; for they cannot, unleſs they meet each other in two oppoſite Points in a Semicircle.

PROPOSITION II.

If from the Pole C *of any Circle* AFB, *be drawn a Right Line* CD *to the Centre thereof, the ſaid Line will be perpendicular to the Plane of that Circle.* Vid. Fig. to Prop. 1,

LET there be drawn any Diameter EF, GH, in the Circle AFB; then, becauſe the Triangles CDF, CDE, the Sides CD, DF, are equal to the Sides CD, DE, and the Baſe CF equal to the Baſe CE (*by Def.* 2.); then (*by* 4. *El.* 1.) ſhall the Angle CDF = Angle CDE, and ſo each of them will be a Right Angle. After the ſame manner we demonſtrate, that the Angles CDG, CDH, are Right Angles; and ſo (*by* 4. *El.* 11.) CD ſhall be perpendicular to the Plane of the Circle AFE. W. W. D.

Coroll. 1. A great Circle is diſtant from its Pole by the Interval of a Quadrant; for ſince the Angles CDG, CDF, are Right Angles, the Meaſures of them, *viz.* the Arcs CG, CF, will be Quadrants.

2. Great Circles, that paſs thro' the Pole of ſome other Circle, make Right Angles with it; and contrariwiſe, if great Circles make Right Angles with ſome other Circle, they ſhall paſs thro' the Poles of that other Circle; for they muſt neceſſarily paſs thro' the Right Line DC.

PROPOSITION III.

If a great Circle ECF *be deſcribed about the Pole* A; *then the Arc* CF *intercepted between* AC, AF, *is the Meaſure of the Angle* CAF, *or* CDF. Vid. Fig. to Prop. 1.

THE Arcs AC, AF, (*by Cor.* 1. *Prop.* 2.) are Quadrants; and conſequently the Angles ADC, ADF, are Right Angles. Wherefore (*by Def.* 6. *El.* 11.) the Angle CDF (whoſe Meaſure is the Arc CF) is equal to the Inclination of the Planes ACB, AFB, and alſo equal to the ſpherical Angle CAF, or CBF. W. W. D.

Coroll. 1. If the Arcs AC, AF, are Quadrants, then ſhall A be the Pole of the Circle paſſing thro' the Points C and F; for AD is at Right Angles to the Plane FDC (*by* 14. *El.* 11.).

2. The vertical Angles are equal, for each of them is equal to the Inclination of the Circles; alſo the adjoining Angles are equal to two Right Angles.

PROPOSITION IV.

Triangles ſhall be equal and congruous, if they have two Sides equal to two Sides, and the Angles comprehending the two Sides alſo equal.

PROPOSITION V.

Alſo Triangles ſhall be equal and congruous, if one Side, together with the adjacent Angles in one Triangle, be equal to one Side, and the adjacent Angles of the other Triangle.

PROPOSITION VI.

Triangles mutually Equilateral, are alſo mutually Equiangular.

PROPOSITION VII.

In Iſoſceles Triangles, the Angles at the Baſe are equal.

PRO-

PROPOSITION VIII.

And if the Angles at the Baſe be equal, then the Triangle ſhall be Iſoſceles.

Theſe four laſt Propoſitions are demonſtrated in the ſame manner, as in plain Triangles.

PROPOSITION IX.

Any two Sides of a Triangle are greater than the third.

FOR the Arc of a great Circle is the ſhorteſt Way, between any two Points in the Superficies of the Sphere.

PROPOSITION X.

A Side of a ſpherical Triangle is leſs than a Semicircle.

LET AC, AB, the Sides of the Triangle ABC, be produced till they meet in D; then ſhall the Arc ACD, which is greater than the Arc AC, be a Semicircle.

PROPOSITION XI.

The three Sides of a ſpherical Triangle are leſs than a whole Circle.

FOR BD+DC is greater than BC (*by Prop. 9.*); and adding on each Side BA+AC, DBA+DCA; that is, a whole Circle will be greater than AB+BC+AC, which are the three Sides of the ſpherical Triangle ABC.

PROPOSITION XII.

In any ſpherical Triangle ABC, *the greater Angle* A *is ſubtended by the greater Side.*

MAKE the Angle BAD=Angle B; then ſhall AD=BD (*by* 8 *of this*); and ſo BDC=DA+DC, and theſe Arcs are greater than AC. Wherefore the Side BC, that ſubtends the Angle BAC, is greater than the Side AC, that ſubtends the Angle B.

PROPOSITION XIII.

In any ſpherical Triangle ABC, *if the Sum of the Legs* AB *and* BC *be greater, equal, or leſs, than a Semicircle, then the internal Angle at the Baſe* AC *ſhall be greater, equal, or leſs, than the external and oppoſite Angle* BCD; *and ſo the Sum of the Angles* A *and* ACB *ſhall alſo be greater, equal, or leſs, than two Right Angles.*

FIRST, let AB + BC = Semicircle = AD, then ſhall BC = BD, and the Angles BCD and D equal, (*by* 8 *of this*) and therefore the Angle BCD ſhall be = Angle A.

Secondly, Let AB + BC be greater than ABD; then ſhall BC be greater than BD; and ſo the Angle D (that is, the Angle A, by 12 of this) ſhall be greater than the Angle BCD. In like manner we demonſtrate, if AB + BC be together leſs than a Semicircle, that the Angle A will be leſs than the Angle BCD: And becauſe the Angles BCD and BCA are = two Right Angles; if the Angle A be greater than the Angle BCD, then ſhall A and BCA be greater than two Right Angles; if the Angle A = BCD, then ſhall A and BCA be equal to two Right Angles. And if A be leſs than BCD, then will A and BCA be leſs than two Right Angles. W.W.D.

PROPOSITION XIV.

In any ſpherical Triangle GHD, *the Poles of the Sides, being joined by great Circles, do conſtitute another Triangle* XMN, *which is the Supplement of the Triangle* GHD; viz. *the Sides* NX, XM, *and* NM, *ſhall be Supplements of the Arcs that are the Meaſures of the Angles* D, G, H, *to the Semicircles; and the Arcs that are the Meaſures of the Angles* M, X, N, *will be the Supplements of the Sides* GH, GD, *and* HD, *to Semicircles.*

FROM the Poles G, H, D, let the great Circles XCAM, TMNO, XKBN, be deſcribed; then, becauſe G is the Pole of the Circle X C A M, we ſhall have G M = Quadrant; (*by Cor.* 1. *Prop* 2.) and ſince H is the Pole of the Circle T M O, then will H M be alſo a Quadrant; and ſo (*by Cor.* 1. *Prop.* 3.) M ſhall be the Pole of the Circle G H. In like manner, becauſe D is the Pole of the Circle X B N, and H the Pole of the Circle T M N, the Arcs D N, H N, will be Quadrants; and ſo (*by Cor.* 1. *Prop.* 3.) N ſhall be the Pole of the Circle H D. And becauſe G X, D X, are Quadrants, X will be the Pole of the Circle G D. Theſe Things premiſed,

Becauſe N K = Quadrant, (*by Cor.* 1. *Prop.* 2.) then will N K + X B, that is, N X + K B = two Quadrants, or a Semicircle; and ſo N X is the Supplement of the Arc K B, or of the Meaſure of the Angle H D G to a Semicircle. In like manner, becauſe M C = Quadrant, and X A = Quadrant, then will M C + X A; that is, X M + A C = two Quadrants or Semicircle; and conſequently X M is the Supplement of the Arc A C, which is the Meaſure of the Angle H G D. Likewiſe, ſince M O, N T, are Quadrants, we ſhall have M O + N T = O T + N M = Semicircle. And therefore N M is the Supplement of the Arc O T, or of the Meaſure of the Angle G H D, to a Semicircle. W. W. D.

Moreover, becauſe D K, H T, are Quadrants, D K + H T, or K T + H D, are equal to two Quadrants, or a Semicircle. Therefore K T, or the Meaſure of the Angle X N M, is the Supplement of the Side H D to a Semicircle. After the ſame manner it is demonſtrated, that O C, the Meaſure of the Angle X M N, is the Supplement of the Side G H; and B A the Meaſure of the Angle X, is the Supplement of the Side G D. W. W. D.

PROPOSITION XV.

Equiangular ſpherical Triangles are alſo equilateral

FOR their Supplementals (*by* 14 *of this*) are equilateral, and therefore equiangular alſo; and ſo themſelves are likewiſe equilateral. (*by Part* 2. *Prop.* 14.)

PROPOSITION XVI.

The three Angles of a ſpherical Triangle are greater than two Right Angles, and leſs than ſix.

FOR the three Meaſures of the Angles G, H, D, together with the three Sides of the Triangle XNM, make three Semicircles (*by* 14 *of this*); but the three Sides of the Triangle XNM are leſs than two Semicircles (*by* 11 *of this*). Wherefore the three Meaſures of the Angles G, H, D, are greater than a Semicircle; and ſo the Angles G, H, D, are greater than two Right Angles.

The ſecond Part of the Propoſition is manifeſt; for in every ſpherical Triangle, the external and internal Angles together, only make ſix Right Angles; wherefore the internal Angles are leſs than ſix Right Angles.

PROPOSITION XVII.

If from the Point R, *not being the Pole of the Circle* AFBE, *there fall the Arcs* RA, RB, RG, RV, *of great Circles to the Circumference of that Circle; then the greateſt of thoſe Arcs is* RA, *which paſſes thro' the Pole* C *thereof; and the Remainder of it is the leaſt; and thoſe that are more diſtant from the greateſt are leſs than thoſe which are nearer to it, and they make an obtuſe Angle with the former Circle* AFB, *on the Side next to the greateſt Arc.* Vid. Fig. to Prop. 1.

BEcauſe C is the Pole of the Circle AFB, then ſhall CD and RS, which is parallel thereto, be perpendicular to the Plane AFB. And if SA, SG, SV, be drawn, then ſhall SA (*by* 7. *El.* 3.) be greater than SG, and SG greater than SV. Whence in the Right-angled plain Triangles RSA, RSG, RSV, we ſhall have RSq+SAq, or RAq, greater than RSq+SGq, or RSq; and ſo RA will be greater than RG, and the Arc RA greater than the Arc RG. In like manner, RSq+SGq, or RGq ſhall be greater than R S q+SVq, or RVq; and ſo R G ſhall be greater than R V, and the Arc R G greater than the Arc R V.

2dly. The Angle R G A is greater than the Angle C G A, which is a Right Angle, (*by Cor. Prop.* 3.) and the Angle R V A is greater than the Angle C V A, which alſo is a Right Angle. Therefore the Angles R G A, R V A, are obtuſe Angles.

PROPOSITION XVIII.

In a ſpherical Triangle right-angled at A, *the Legs containing the Right Angle, are of the ſame Affection with the oppoſite Angles; that is, if the Legs be greater or leſs than Quadrants, then accordingly will the Angles oppoſite to them be greater or leſs than Right Angles.* Vid. Fig. to Prop. 1.

FOR if A C be a Quadrant, then will C be the Pole of the Circle A F B, and the Angles A G C, or A V C, will be Right Angles. If the Leg A R be greater than a Quadrant, then ſhall the Angle A G R be greater than a Right Angle (*by* 17 *of this*); and if the Leg A X be leſs than a Quadrant, the Angle A G X ſhall be leſs than a Right Angle.

PROPOSITION XIX.

If two Legs of a right-angled ſpherical Triangle be of the ſame Affection, (and conſequently the Angles) that is, if they are both leſs or both greater than a Quadrant, then will the Hypothenuſe be leſs than a Quadrant. Vid. Fig. to Prop. 1.

IN the Triangle A R V, or B R V, let F be the Pole of the Leg A R; then will R F be a Quadrant, which is greater than R V (*by* 17 *of this*).

PROPOSITION XX.

If they be of a different Affection, then ſhall the Hypothenuſe be greater than a Quadrant. Vid Fig. to Prop. 1.

FOR in the Triangle A R G, the Hypothenuſe R G is greater than R F, which is a Quadrant.

PROPOSITION XXI.

If the Hypothenuſe be greater than a Quadrant, then the Legs of the Right Angle, and ſo the Angles oppoſite to them, are of a different Affection; but if leſſer, of the ſame Affection. Vid. Fig. to Prop. 1.

THIS Propoſition being the Converſe of the former ones, eaſily follow from them.

PROPOSITION XXII.

In any ſpherical Triangle ABC, *if the Angles at the Baſe* B *and* C, *be of the ſame Affection, then the Perpendicular falls within the Triangle; and if they be of a different Affection, the Perpendicular falls without the Triangle.*

IN the firſt Caſe, if the Perpendicular does not fall within, let it fall without the Triangle, (as in *Fig.* 2.) then in the Triangle ABP, the Side AP is of the ſame Affection with the Angle B. And in like manner, in the Triangle ACP, AP is of the ſame Affection with the Angle ACP. Therefore, ſince ABC, and ACP, are of the ſame Affection, the Angles ABC, ACB, ſhall be of a different Affection; which is contrary to the Hypotheſis.

In the ſecond Caſe, if the Perpendicular does not fall without, let it fall within. (as in *Fig.* 1.) Then in the Triangle ABP, the Angle B is of the ſame Affection with the Leg AP. So likewiſe, in the Triangle ACP, the Angle C is of the ſame Affection with AP; and therefore the Angles B and C are of the ſame Affection; which is contrary to the Hypotheſis.

PROPOSITION XXIII.

In ſpherical Triangles BAC, BHE, *right-angled at* A *and* H, *if the ſame acute Angle* B *be at the Baſe* BA, *or* BH, *then the Sines of the Hypothenuſes ſhall be proportional to the Sines of the perpendicular Arcs.*

FOR the Right Lines CD, EF, being perpendicular to the ſame Plane, are parallel. Alſo FR, DP, perpendicular to the Radius OB, are likewiſe parallel; wherefore the Planes of the Triangles EFR, CDP, are alſo parallel. (*by* 15. *El.* 11.) Wherefore CP, ER, the common Sections of thoſe Planes, with the Plane paſſing thro' BE, CO, will be parallel. (*by* 16. *El.* 11.) Therefore the Triangles CDP, EFR, ſhall be equiangular. Wherefore CP the Sine of the Hypothenuſe BC, is to CD the Sine of the perpendicular Arc CA, as ER the Sine of the Hypothenuſe BE, to EF the Sine of the perpendicular Arc EH. W.W.D.

PROPOSITION XXIV.

The ſame Things being ſuppoſed, AQ, HK, *the Sines of the Baſes, are proportional to* IA, GH, *the Tangents of the perpendicular Arcs.*

FOR after the ſame manner, as in the laſt Propoſition, we demonſtrate that the Triangles QAI, KHG, are equiangular; whence QA : AI :: KH : HG.

PROPOSITION XXV.

In a ſpherical Triangle ABC, *right-angled at* A, *as the Coſine of the Angle* B, *at the Baſe* BA, *is to the Sine of the vertical Angle* ACB, *ſo is the Coſine of the Perpendicular to the Radius.*

PREPARATION.

LET the Sides AB, BC, CA, be produced, ſo that BE, BF, CI, CH, be Quadrants; and from the Poles B and C, draw the great Circles EFDG, IHG; then will the Angles at E, F, I, H, be Right Angles. And ſo D is the Pole of BAE, (*by Cor.* 2. *Prop.* 2. *of this*) and G the Pole of IFCB; alſo AE will be = Complement of the Arc BA, and FE the Meaſure of the Angle B = GD, and DF their Complement: Alſo BC ſhall be = FI = Meaſure of the Angle G, and CF their Complement. Likewiſe CA = HD, and DC their Complement. Theſe Things premiſed in the Triangles HIC, DCF, right-angled at I and F, and having the ſame acute Angle C, ſince BA is leſs than a Quadrant, it will be as S. DF:

DF : S, HI : : S, DC : S, HC; that is, the Coſine of the Angle B is to the Sine of the vertical Angle BCA, as the Coſine of CA is to Radius. W. W. D.

PROPOSITION XXVI.

The Coſine of the Baſe : *Coſine of the Hypothenuſe* : : R : *Coſ. of the Perpendicular.*

FOR in the Triangles AED, CFD, right-angled at E, F, having the ſame acute Angle D; becauſe AE is leſs than a Quadrant, we have S, EA : S, CF : : S, DA : S, DC. W. W. D.

PROPOSITION XXVII.

S, *of the Baſe* : R : : T, *of the Perpendicular* : T, *of the Angle at the Baſe.*

FOR in the Triangles BAC, BEF, right-angled at A and E, and having the ſame acute Angle B; becauſe AC is leſs than a Quadrant, we have S, BA : S, BE : : T, AC : T, EF. W. W. D.

PROPOSITION XXVIII.

Coſ. of the vertical Angle : R : : T, *of the Perpendicular* : T, *of the Hypothenuſe.*

IN the Triangles GIF, GHD, right-angled at I and H, and having the ſame acute Angle G, becauſe HD is leſs than HC, or a Quadrant, it is as S, GH : S, GI : : T, HD : T, IF.

PROPOSITION XXIX.

S, *of the Hypothenuſe* : R : : S, *of the Perpendicular* : S, *of the Angle at the Baſe.*

IN the aforeſaid Triangles, we have S, IF : S, GF : : S, HD : S, GD.

PROPOSITION XXX.

R : *Coſ. of the Hypothenuſe* : : T, *of the vertical Angle* : *Cot. of the Angle at the Baſe.*

IN the Triangles HIC, DFC, right-angled at I and F, and having the ſame acute Angle C, becauſe DF is leſs than a Quadrant, we have S, CI : S, CF : : T, HI : T, DF; that is, R : Coſ. BC : : T, C : Cot. B.

The laſt ſix Propoſitions are ſufficient for ſolving all the ſixteen Caſes of Right-angled ſpherical Triangles. Theſe ſixteen Caſes, with their Analogies deduc'd from the ſaid Propoſitions, are as follow.

	Given beſides the Right Angle.	Sought	*Vid. Fig.* to *Prop.* 25, 26, 27, 28, 29, 30.	
1	AC and C.	B.	R : Coſ. CA : : S, C : Coſ. B of the ſame Kind with CA.	By the Inverſe of Prop. 25.
2	AC and B.	C.	Coſ. CA : R : : Coſ. B : S, C, this is ambiguous.	by Prop. 25.
3	B and C.	AC.	S, C : Coſ. B : : R : Coſ. CA, of the ſame Kind with the Angle B.	by Prop. 25, and 18.
4	BA, CA.	BC.	R : Coſ. BA : : Coſ. AC : Coſ. BC. If BA, AC, be of the ſame Affection, and not Quadrants, then BC will be leſs than a Quadrant. If they be of a different Affection, BC ſhall be greater than a Quadrant.	by Prop. 26, 19, and 20.

Given

	Given beſides the Right Angle.	Sought		
5	B A, B C.	A C.	Coſ. B A : R :: Coſ. B C : Coſ. C A. If B C be leſs than a Quadrant, then ſhall B A and C A be of the ſame Affection, if greater, of a different; but B A is given, and therefore the Species thereof. Wherefore the Species of A C is alſo given.	By Prop. 26, and 21.
6	B A, C A.	B.	S, B A : R :: T, C A : T, B of the ſame Affection with the oppoſite Side C A.	by Prop. 27, and 18.
7	BA, B.	A C.	R : S, B A :: T, B : T, A C of the ſame Kind with B.	by Prop. 27, and 18.
8	AC,B.	B A.	T, B : R :: T, C A : S, B A ambiguous.	by Prop. 27.
9	BC,C.	A C.	R : Coſ. C :: T, BC : T, CA. If B C be leſs than a Quadrant, the Angles C and B are of the ſame Affection; if greater, of a different. Therefore, if the Species of the Angle B be given, then will A C be given.	by Prop. 28, and 21.
10	AC,C.	B C.	Coſ. C : R :: T, C A : T, B C. And ſo, if the Angle C, and C A, be of the ſame Affection, then B C ſhall be leſſer than a Quadrant; if of a different, greater.	by Prop. 28, 20, 21.
11	B C, A C.	C.	T, B C : R :: T, C A : Coſ. C. If B C be leſs than a Quadrant, then C A and B A, and conſequently the Angles, ſhall be of the ſame Affection; if greater, of a different, but the Species of C A is given. Therefore the Species of the Angle C will be alſo given.	by Prop. 28, and 21.

	Given beſides the Right Angle.	Sought		
12	BC,B.	AC.	R : S, BC :: S, B : S, AC of the ſame Species with B.	By Prop. 29, and 18.
13	AC,B.	BC.	S, B : S, AC :: R : S, BC ambiguous.	by Prop. 29.
14	BC. AC.	B.	S, BC : R :: S, AC : S, B of the ſame Species with CA.	by Prop. 29.
15	B,C.	BC.	T, C : R :: Cot. B : Coſ. BC. And ſo, if the Angles B and C are of the ſame Affection, then ſhall BC be leſs than a Quadrant; if of a different, greater.	by Prop. 30, 19, and 20.
16	BC,C	B.	R : Coſ. BC :: T, C : Cot. B. And ſo if BC be leſſer than a Quadrant, the Angles C and B ſhall be of the ſame Affection; if greater, of a different. But the Species of the Angle C is given; therefore the Species of the Angle B will be given alſo.	by Prop. 30, and 31.

Of the Solution of Right-angled ſpherical Triangles, by the five circular Parts.

THE Lord *Napier*, (the noble Inventor of Logarithms) by a due Conſideration of the Analogies, by which right-angled ſpherical Triangles are ſolv'd, found out two Rules, eaſy to be remembered, by means of which, all the ſixteen Caſes may be ſolv'd; for ſince in theſe Triangles, beſides the Right Angle, there are three Sides, and two Angles; the two Sides comprehending the Right Angle, and the Complements of the Hypothenuſe, and the two other Angles, were called by *Napier*, *Circular Parts.* And then there are given any two of the ſaid Parts, and a third is ſought; one of theſe three, which is called the *Middle Part*, either lies between the other two Parts, which are called *Adjacent Extremes*; or is ſeparated from them, and then are called *Oppoſite Extremes*; ſo if the Complement of the Angle B (*Fig. to Prop.* 25.) be ſuppoſed the middle Part, then the Leg AB, and the Complement of the Hypothenuſe BC, are adjacent extreme Parts; but the Complement of the Angle C, and the Side AC, are oppoſite Extremes. Alſo, if the Complement of the Hypothenuſe BC be ſuppoſed the middle Part, then the Complements of the Angles B and C are adjacent Extremes, and the Legs AB, AC, are oppoſite Extremes. In like manner, ſuppoſing the Leg AB the middle Part, the Complement of the Angle B and AC are adjacent Extremes; for the Right Angle A does not interrupt the Adjacence, becauſe it is not a circular Part. But the Complement of the Angle C, and the Complement of the Hypothenuſe BC, are oppoſite Extremes to the ſaid middle Part. Theſe Things premiſed,

RULE I.

In any right-angled ſpherical Triangle, the Rectangle under the Radius, and the Sine of the middle Part, is equal to the Rectangle under the Tangents of the adjacent Parts.

RULE II.

The Rectangle under the Radius, and the Sine of the middle Part, is equal to the Rectangle under the Coſines of the oppoſite Parts.

Each of the Rules have three Caſes. For the middle Part may be the Complement of the Angle B, or C, or the Complement of the Hypothenuſe BC; or one of the Legs, AB, AC.

Caſe 1. Let the Complement of the Angle C be the middle Part. Then ſhall AC, and the Complement of the Hypothenuſe BC, be adjacent Extremes. By *Prop.* 28. the Coſine of the Vertical Angle C is to Radius, as the Tangent of CA is to the Tangent of the Hypothenuſe BC. Then (by Alternation) we ſhall have Coſ. C : T, CA : : R : T, BC. But R : T, BC : : Cot. BC : R (as has been before ſhewn). Wherefore Coſ. C ; T, AC : : Cot. BC : R; whence R × Coſ. C = T, AC × Cot. BC.

And the Complement of the Angle B, and AB, are oppoſite Extremes, to the ſame middle Part, the Complement of the Angle C; and, (*by Prop.* 25.) as the Coſine of the Angle C, to the Sine of the Angle CDF, ſo is the Coſine of DF to Radius. But the Sine of CDF =, AE = Coſ. BA, and Coſ. DF = S, EF = S, Angle B. Whence it will be as Coſ. C : Coſ. BA : : S, B : R. And R × Coſ. C = Coſ. BA × S, B; that is, Radius drawn into the Sine of the middle Part, is equal to the Rectangle under the Coſines of the oppoſite Extremes.

Caſe 2. Let the Complement of the Hypothenuſe BC be the middle Part; then the Complements of the Angles B and C will be adjacent Extremes. In the Triangle DCF (*by Prop.* 27.) it is as S, CF : R : : T, DF ; T, C. Whence (by Alternation) S, CF : T, DF : : (R : T, C : :) Cot. C : R. But S, CF = Coſ. BC and T, DF = Cot. B. Wherefore R × Coſ. BC = Cot. C × Cot. B; that is, Radius drawn into the Sine of the middle Part, is equal to the Product of the Tangents of the adjacent extreme Parts.

And BA, AC, are the oppoſite Extremes to the ſaid middle Part, *viz.* the Complement of BC; and (*by Prop.* 26.) Coſ. BA : Coſ. BC : : R : Coſ. AC Wherefore we ſhall have R × Coſ. BC = Coſ. BA × Coſ. AC.

Caſe 3. Laſtly, let AB be the middle Part; and then the Complement of the Angle B, and AC will be adjacent Extremes, and (*by Prop.* 27.) S, AB : R : : T, CA : : T B. Whence, S, AB : T, CA : : (R : T, B : :) Cot. B : R. And ſo R × S, AB = T, CA × Cot. B.

Moreover, the Complement of BC, and the Angle C, are oppoſite Extremes to the ſame middle Part AB; and in the Triangles GHD (*by Prop.* 25.) we have Coſ. D : S, DGH : : Coſ. GH : R. But Coſ. D = Coſ. AE = S, AB, and S, G = S, IF = S, BC. Alſo Coſ. GH = S, HI = S, C. Wherefore it will be as S, AB : S, BC : : S, C : R. And hence R × S, AB = S, BC × S, C.

And ſo in every caſe the Rectangle under the Radius, and the Sine of the middle Part, ſhall be equal to the Rectangle under the Coſines of the oppoſite Extremes, and to the Rectangle under the Tangents of the adjacent Extremes. And conſequently, if the aforeſaid Equations be reſolved into Analogies, (*by* 16 *El.* 6.) the unknown Parts may be found by the Rule of Proportion. And if the Part ſought be the middle one; then ſhall the firſt Term of the Aanlogy be Radius, and the ſecond and third, the Tangents or Coſines of the extreme Parts. If one of the Extremes be ſought, the Analogy muſt begin with the other; and the Radius, and the Sine of the middle Part, muſt be put in the middle Places, that ſo the Part ſought may be in the fourth Place.

IN oblique-angled ſpherical Triangles (*Fig. to Prop.* 31.) BCD, if a perpendicular Arc AC be let fall from the Angle C to the Baſe, continued, if need be, ſo as to make two Right-angled ſpherical Triangles BAC, DAC; then by thoſe Right-angled Triangles may moſt of the Caſes of oblique-angled ones be ſolved.

PROPOSITION XXXI.

The Coſines of the Angles B *and* D, *at the Baſe* BD, *are proportional to the Sines of the Vertical Angles* BCA, DCA.

FOR Coſ. Angle B : S, BCA : : (Coſ. CA : R : :) Coſ. D : S, DCA (*by* 25. *of this*).

PROPOSITION XXXII.

The Coſines of the Sides BC, DC, *are proportional to the Coſines of the Baſes* BA, DA.

FOR Coſ. BC : Coſ. BA : : (Coſ. CA : R : :) Coſ. DC : Coſ. DA (*by* 26. *of this*).

PROPOSITION XXXIII.

The Sines of the Baſes BA, DA, *are in a reciprocal Proportion of the Tangents of the Angles* B *and* D *at the Baſe* BD.

BEcauſe (*by* 27. *of this*) S, BA : R : : T, AC : T, of the Angle B. And by the ſame inverſely, R : S, DA : : T, of the Angle D : T, AC. Then will it be (by the Equalily of perturbate Ratio, according to *Prop.* 23. *El.* 5.) S, BA : S, DA : : T, Angle D ; T, Angle B.

PROPOSITION XXXIV.

The Tangents of the Sides BC, DC, *are in a reciprocal Proportion of the Coſines of the Vertical Angles* BCA, DCA.

BEcauſe by alternating the 28th *Propoſition*, we have

T, BC : R : : T, CA : Coſ. BCA,

and by the ſame R : Coſ. DCA : : T, DC : T, CA. Wherefore by Equality of perturbate Proportion,

T, BC : Coſ. DCA : : T, DC : Coſ. BCA.

PROPOSITION XXXV.

The Sines of the Sides BC, DC, *are proportional to the Sines of the oppoſite Angles* D *and* B.

BEcauſe by the 29th *of this*, S, BC : R :: S, CA : S, of the Angle B. And by the ſame, inverting R : S, DC :: S, Angle D : S, of CA; whence, by Equality of perturbate Ratio, S, BC : S, DC :: S, D : S, B.

PROPOSITION XXXVI.

In any ſpherical Triangle ABC, *the Rectangle* CF × AE, *or* FM × AE, *contained under the Sines of the Legs*, BC, BA, *is to the Square of the Radius, as* IL *or* IA—LA *the Difference of the verſed Sines of the Baſe* CA, *and the Difference of the Legs* AM, *to* GN, *the verſed Sine of the Angle* B.

LET a great Circle PN be deſcribed from the Pole B; and let BP, BN, be Quadrants; and then PN is the Meaſure of the Angle B; alſo deſcribe from the ſame Pole B a leſſer Circle CFM thro' C; the Planes of theſe Circles ſhall be perpendicular to the Plain BON (*by the 2d of this*). And PG, CH, being perpendicular in the ſame Plain, fall on the common Sections ON, FM; ſuppoſe in G, H. Again draw HI, perpendicular to AO; and then the Plain draw thro' CH, HI, ſhall be perpendicular to the Plain AOB. Whence AI, which is perpendicular to HI, will be perpendicular to the Right Line CI (*by Def.* 4. *El.* 11.); and ſo AI is the verſed Sine of the Arc AC, and AL the verſed Sine of the Arc AM = BM — BA = BC — BA. The Iſoſceles Triangles CFM, PON, are equiangular, ſince MF, NO, as alſo CF, PO, (*by* 16. *El.* 11.) are parallel. Wherefore, if Perpendiculars CH, PG, be drawn to the Sides FM, ON, the Triangles will be divided ſimilarly, and we ſhall have FM : ON :: MH : GN, Alſo, becauſe the Triangles AOE, DIH, DLM, are equiangular, we ſhall have AE : AO :: IL : MH.

But it has been proved, that $FM : ON :: MH : GN$. Wherefore it ſhall be as $AE \times FM : AO \times ON :: IL \times MH \times MH \times GN$, or as IL to GN, that is, the Rectangle, under the Sines of the Legs, is to the Square of Radius, as the Difference of the verſed Sines of the Baſe, and the Difference of the Legs BC, BA, is to the verſed Sine of the Angle B. W. W. D.

PROPOSITION XXXVII.

The Difference of the verſed Sines of two Arcs drawn into half the Radius, is equal to the Rectangle under the Sine of half the Sum and the Sine of half the Difference of thoſe Arcs.

LET there be two Arcs, BE, BF, whoſe Difference EF, let be biſected in D; then ſhall BD be the half Sum, and FD the half Difference of thoſe Arcs. $GE = IL$ is the Difference of the verſed Sines of the Arcs BE, BF; alſo FO is the Sine of the half Differences of the Arcs. And becauſe the Triangles CDK, FEG, are equiangular, we have $DK : GE :: (CD : FE ::) \frac{1}{2} CD : \frac{1}{2} FE$. Whence $DK \times \frac{1}{2} FE$, or $DK \times FO = GE \times \frac{1}{2} CD = IL \times \frac{1}{2} CD$. W. W. D.

PROPOSITION XXXVIII.

The verſed Sine of any Arc, drawn into half the Radius, is equal to the Square of the Sine of one Half of the ſaid Arc.

THE Triangles CBM, DEB, are equiangular, ſince the Angles at M and E are Right Angles, and the Angle at B is common. Wherefore $EB : BD :: BM : BC$. And then will $EB \times BC = BM \times BD$; and $EB \times \frac{1}{2} BC = BM \times \frac{1}{2} BD = BMq$. W. W. D.

PROPOSITION XXXIX.

In any ſpherical Triangle ABC, *whoſe Legs containing the Angle* B *are* BC, AB, *and Baſe ſubtending that Angle* AC: *If the Arc* AM *be taken* $=$ *Difference of the Legs* $= BC - AB$. *Then ſhall the Rectangle under the Sines of the Legs* BC, BA, *be to the Square of the Radius, as the Rectangle under the Sine of the Arc* $\frac{AC+AM}{2}$ *and the Sine of the Arc* $\frac{AC-AM}{2}$ *is to the Square of the Sine of one Half of the Angle* B. Vid. Fig. to Prop. 36.

BEcauſe the Rectangle under the Sines of the Legs AB, BC, is to the Square of Radius, as IL is to the verſed Sine of the Angle B, or as $R \times IL$ to $\frac{1}{2}R$ drawn in the verſed Sine of the Angle B (*by Prop.* 36. *of this*). And ſince $\frac{1}{2}R \times IL =$ Rectangle under the Sines of the Arcs $\frac{AC+AM}{2}$, and $\frac{AC-AM}{2}$ (*by Prop.* 37. *of this*). And alſo $\frac{1}{2}R$ drawn into the verſed Sine of the Angle B is equal to the Square of the Sine one Half of the Angle B. Therefore the Rectangle under the Sines of the Sides to the Square of Radius, ſhall be as the Rectangle under the Sines of the Arcs $\frac{AC+AM}{2}$ and $\frac{AC-AM}{2}$, is to the Square of the Sine of one Half the Angle B. W. W. D.

The Twelve Caſes of oblique-angled ſpherical Triangles are as follow.

	Given	Sought	Make as
1	Angles B, D, and BC.	Angle C.	R : Coſ. BC : : T, B : Cot. BCA (*by Prop.* 30. *of this*): Alſo Coſ. B. : S, BCA : : Coſ. D : S, DCA (*by* 31. *of this*). Wherefore the Sum of the Angles BCA, DCA, if the Perpendicular falls within the Triangle, or the Difference, if it falls without, will be = BCD. Whether the Perpendicular falls within or without the Triangle, may be known from the Affection of the Angles B and D (*by* 22. *of this*); which Admonition ought to be obſerved in the following Solutions.
2	Angles B, BCD and the Side BC.	Angle D.	R : Coſ. BC : : T, B : Cot. BCA (*Prop.* 30. *of this*). And S, BCA : S, DCA : : Coſ. B : Coſ. D (*by Prop.* 31.). If BCA be leſs than BCD, the Angle D ſhall be of the ſame Affection with the Angle B. If BCA be greater than the Angle BCD, then the Angles B and D ſhall be of a different Affection, by the Converſe of *Prop.* 22.
3	The Sides BC, CD and the Angle B.	The Side BD.	R : Coſ. B : : T, BC : T, BA (*by* 28. *of this*). And Coſ. BC : : Coſ. BA : : Coſ. DC : Coſ. DA (*by* 32. *of this*). The Sum or Difference of BA and DA, according as the Perpendicular falls within or without the Triangle, is equal to BD; which cannot be known, unleſs the Species of the Angle D be firſt known.

In the Original, the Proportion was thus; Coſ. BC : R : : TB : Cot. BCA.

This Proportion in the Original was as in the foregoing. The Species of the Angle BCA *may be known by* Prop. 18. *and* 19.

	Given	Sought	Make as
4	The Sides BC, DB, and the Angle B.	The Side CD.	R : Coſ. B :: T, BC : T, BA (by 28. of this). And Coſ. BA : Coſ. BC :: Coſ. DA : Coſ. DC (by *Prop.* 32. of this.) According as DA is ſimilar or diſſimilar to CA, or to the Angle BDC, ſo ſhall DC be leſſer or greater than a Quadrant (by 19. and 20. of this).
5	Angle B, D, and the Side BC.	The Side BD.	R : Coſ. B :: T, BC : T, BA (by 28. of this). And T, D : T, B :: S, BA : S, DA (by 33. of this). The Sum or Difference of which BA and DA is = BD.
6	The Sides BC, BD, and the Angle B.	Angle D.	R : Coſ. B :: T, BC : T, BA (by *Prop.* 28. of this.) And S, DA : S, BA :: T, B : T, D (by 33. of this). According as BD is greater or leſſer than BA, the Angle D ſhall be ſimilar or diſſimilar to the Angle B (by 22. of this).
7	The Sides BC, DC, and the Angle B.	Angle C.	Coſ. BC : R :: Cot. B : T, BCA, (by 30. of this). And T, DC : T, BC :: Coſ. BCA : Coſ. DCA (by 34. of this). The Sum or Difference of the Angles BCA, DCA, according as the Perpendicular falls within or without the Triangle, is equal to the Angle BCD.
8	The Angles BCD, and B, and the Side BC.	The Side DC.	Coſ. BC :: R :: Cot. B : T, BCA (by 30. of this). Alſo Coſ. DCA : Coſ. BCA :: T, BC : T, DC (by 34. of this). If the Angle DCA be ſimilar to the Angle B, (that is, if AD be ſimilar to CA) then DC ſhall be leſs than a Quadrant. If the Angles DCA and B be diſſimilar, then DC ſhall be greater than a Quadrant, which follows (from *Prop.* 18, 19, and 20 of this).

	Given	Sought	Make as
9	The Sides BC, DC, and the Angle B.	The Angle D.	S, CD : S, B : : S, BC : S, D : which is ambiguous. The Analogy follows from *Prop.* 35. of this.
10	The Angles B, D, and the Side BC.	The Side DC.	S, D : S, BC : : S, B : S, DC, which Side is ambiguous.
11	All the Sides AB, BC, CA. *Vid. Fig. Prop.* 36.	The Angle B.	As the Rectangle under the Sines of the Legs AB, BC : the Square of Radius : : the Rectangle under the Sines of the Arcs $\frac{AC+AM}{2}$ and $\frac{AC-AM}{2}$: the Square of the Sine of $\frac{1}{2}$ the Angle B (by *Prop.* 39.).
12	All the Angles G, H, D. *Vid. Fig. Prop.* 14.	The Side GD.	In the Triangle XNM, the Arc MN is the Complement of the Angle GHD to a Semicircle. XM is the Complement of the Angle G, and XN, the Complement of the Angle D. And the Angle X, the Complement of the Side GD to a Semicircle. Wherefore, if the Angles be changed into Sides, and the Sides into Angles, the Operation will be the ſame, as in *Caſe* 11. of this, ſince Arcs and their Complements to Semicircles have the ſame Sines.

The following REMARK *by* SAMUEL CUNN.

THAT this is true but in a particular Caſe, *viz.* when two of the Angles of the Triangle are Right ones, and two of the Sides Quadrants, may be thus demonſtrated. For if poſſible, let ſome Triangle RST, *Fig.* to *Prop.* 14th, be ſuch, that its Sides RS, ST, TR, be equal to the Meaſures of GHD, HGD, GDH, the Angles of a Triangle GHD; and alſo, that the Meaſures of RST, STR, TRS, the Angles of the Triangle RST, be equal to GH, GD, HD, the Sides of the Triangle GHD. And produce MX, MN, two Sides of the ſupplemental Triangle, to Semicircles, and they will meet ſomewhere; ſuppoſe at E; and there will be conſtructed thereby the Triangle NEX, of which XE (the Supplement of XM, which, by the 14*th Prop.* was the Supplement of the Meaſure of the Angle HGD) is equal to the Meaſure itſelf of the ſame Angle HGD: And in like manner, NE, the Supplement of NM, which, by the 14*th Prop.* was the Supplement of the Meaſure of the Angle GHD, is equal to the Meaſure itſelf of the ſame Angle GHD. But the third Side XN is not the Meaſure of the third Angle GDH, but its Supplement, by the 14*th Prop.* Moreover, of the Angle EXN, (whoſe Supplement is NXM) the Meaſure, by the 14*th Prop.* is equal to GD; and of the Angle XNE, (whoſe Supplement is MNX) the Meaſure, by the 14*th Prop.* is equal to HD. But of the third NEX, (which is equal to NMX) the Meaſure is not equal to GH, but its Supplement.

Now make NV=RT=BK, the Meaſure of the Angle GDH, and draw the great Circle EV. And ſince RS, by Suppoſition, is equal to the Meaſure of the Angle GHD, which is equal to EN; and ſince the Meaſure of the Angle SRT is, by Suppoſition, equal to DH, which is alſo equal to the Meaſure of the Angle XNE; the Angle XNE is equal to the Angle R. Then conſequently, by the

4*th Prop*. the Triangles S R T, E N V, will have the Baſe S T, equal to the Baſe E V; the Angle T, to the Angle N V E, and the Angle S, to the Angle N E V. But S T, (which is equal to E V) by Suppoſition, is equal to the Meaſure of the Angle H G D; to which Meaſure X E is alſo equal. Therefore E V is equal to X E; and conſequently, by the 7*th Prop*. the Angle E V X is equal to the Angle E X V; and the Angle E X V (whoſe Meaſure, as hath been ſhewn above, is equal to G D) is equal to the Angle T, (or N V E) ſince, by Suppoſition, the Meaſure of this is alſo equal to G D. Therefore the Angle E V X is equal to the Angle E V N, and ſo both right ones; and conſequently E X V a right one alſo. Therefore, by the 2*d Cor.* to the 2*d Prop.* E V and E X are both Quadrants.

But if E V be a Quadrant, and at Right Angles to N X, then E, by 2*d Prop.* and its *Coroll.* is the Pole of N X; and ſo E N a Quadrant alſo, and the Angle E N V a right one. Therefore, if the Sides of a Triangle (N E V, or its Equal) R S T are equal to the Meaſures of the Angles of ſome other Triangle G H D, and the Meaſures of the Angles of the former, equal to the Sides of the latter; two Sides of ſuch a Triangle R S T, or G H D, muſt be Quadrants, and two Angles of each Right ones.

Therefore, if a Triangle R S T be conſtructed, whoſe Sides are equal to the Meaſure of the Angles of another Triangle G H D; the Meaſures of the Angles of the Triangle R S T ſhall not be equal to the Sides of the Triangle GHD, unleſs in the one Caſe before-mentioned. Therefore the Meaſures of the Angles of the Triangle G H D, uſed as the Sides of a Triangle in the 11*th Caſe*, will not give us a Side of G H D, but the Meaſure of an Angle of the Triangle R S T, unleſs in the one afore-mention'd Caſe; *which was to be demonſtrated.*

But to find a Side GD of a ſpherical Triangle GHD, whoſe Angles are all given, produce M N, that Side of the ſupplemental Triangle, which is equal to the Supplement of the Meaſure of G H D, the Angle oppoſite to the Side ſought, and M X, either of the other Sides till they meet as in E. And there, as hath been before ſhewn, the Sides E X, E N, of the Triangle

angle E X N, are exactly equal to the Meaſures of the Angles H G D, G H D, of the Triangle G H D; and of the Angles E X N, E N X, of the Triangle E X N, the Meaſures are equal to G D, H D. But the Side X N is equal to the Supplement of the Meaſure of the Angle G D H. And of the Angle X E N, the Meaſure is equal to the Supplement of G H.

Therefore the SOLUTION is thus:

Change one of the Angles G D H, adjacent to the Side ſought, into its Supplement; and then work with the Meaſures of the Angles as tho' they were Sides, and the Reſult will be G D, the Side ſought.

The preceding Fault, as well as the Omiſſions hereafter mentioned, are not peculiar to our Author; but may be found in Dr. *Harris*, Mr. *Caſwell*, Mr. *Heynes*, and many other Trigonometrical Writers.

In the Solution of our 8th and 9th Caſes, they have told us, that the *Quæſita* are ambiguous; which ſometimes, indeed, is true, but ſometimes alſo falſe: Therefore, as I conceive it, they ought to have laid down Rules, by Help of which we might diſcover when the *Quæſita* are ambiguous, and when not.

This Overſight may be corrected by the following Directions: Wherein, becauſe every Sine correſponds to two Arches, to one leſs than a Quadrant, and to another, which is the Supplement of the former to a Semicircle, (a true Diſtinction of which of theſe are to be uſed, being neceſſary to be known, before a proper Solution can be given to ſuch Problems as theſe are) I ſhall beg Leave, for Brevity-ſake, to call the leſſer Arch the Acute Value, and the greater the Obtuſe; whether the Sine be of an Angle or a Side.

In the tenth Caſe, there are given two Angles B, D, *and* B C, *a Side oppoſite to one of thoſe Angles* D, *to find* D C *the Side oppoſite to the other.*

TO the acute Value of D C, and alſo to its obtuſe one, add B C; and if each of theſe Sums are greater

$\left\{\begin{matrix}\text{greater}\\ \text{leſs}\end{matrix}\right\}$ than a Semicircle, when the Sum of the Angles B, D, is $\left\{\begin{matrix}\text{greater}\\ \text{leſs}\end{matrix}\right\}$ than two Right Angles; both the Values of DC may be admitted, and then is ambiguous: But when only one of thoſe Sums is $\left\{\begin{matrix}\text{greater}\\ \text{leſs}\end{matrix}\right\}$ than a Semicircle, only one Value of DC can be true, *viz.* the $\left\{\begin{matrix}\text{obtuſe}\\ \text{acute}\end{matrix}\right\}$ one; and then is not ambiguous.

In the ninth Caſe, there are given two Sides BC, DC, *and one Angle* B, *oppoſite to* DC *one of thoſe Sides, to find* D *the Angle oppoſite to the other.*

TO the acute Value of D, and alſo to its obtuſe Value, add B; and if each of theſe Sums is $\left\{\begin{matrix}\text{greater}\\ \text{leſs}\end{matrix}\right\}$ than two Right Angles, when the Sum of the Sides is $\left\{\begin{matrix}\text{greater}\\ \text{leſs}\end{matrix}\right\}$ than a Semicircle, both the Values of D may be admitted, and conſequently D is ambiguous: But when only one of thoſe Sums is $\left\{\begin{matrix}\text{greater}\\ \text{leſs}\end{matrix}\right\}$ than two Right Angles, only one Value of D is true, *viz.* the $\left\{\begin{matrix}\text{obtuſe}\\ \text{acute}\end{matrix}\right\}$ one; and then not ambiguous.

Nor are we better uſed in the firſt Propoſition; for tho' it is determined by the given Angles, whether the Perpendicular falls within or without the Triangles, yet in each of thoſe Varieties, the *Quæſita* will be ſometimes ambiguous, and ſometimes not.

In this firſt Propoſition there are given two Angles B, D, *and* BC, *a Side oppoſite to* D, *one of them, to find* C *the third Angle.*

1. Let the Perpendicular fall within; that is, let the given Angles be of the ſame Species.

TO the acute Value of DCA, and alſo to its obtuſe one, add the Angle BCA; and if each of theſe Sums is leſs than two Right Angles, then either the acute Value of DCA, or its obtuſe one added to BCA, gives a Value of BCD; which, therefore, is ambiguous. And when only one of theſe Sums is leſs than two Right Angles, the acute Value of DCA, added to BCA, gives the only Value of BCD; which then is not ambiguous, though in both Varieties the Perpendicular fell within.

2. Let the Perpendicular fall without; that is, let the given Angles be of different Species.

WHEN the obtuſe Value of the Angle DCA is leſs than the Angle BCA, the Angle BCD may be had by ſubtracting either Value of DCA from BCA; and then BCD is ambiguous. But when the obtuſe Value of DCA is not leſs than BCA, the acute Value of DCA, taken from BCA, gives the ſingle Value of BCD; which, therefore, is not ambiguous; tho' in both Varieties the Perpendicular fell without.

In the fifth Caſe we lie under the ſame Misfortune, where there are given, as in the firſt, the Angles B, D, *and the Side* BC, *to find* BD *the Side lying between thoſe given Angles.*

1. When the Perpendicular falls within; that is, when the given Angles are of the ſame Species.

TO the acute Value of DA, and ſo alſo to its obtuſe one, add BA; and if each of theſe Sums is leſs than a Semicircle, then either the acute Value of DA, or its obtuſe one, added to BA, gives the

Value of BD; which thence is ambiguous. And when only one of theſe Sums is leſs than a Semicircle, the acute Value of DA, added to BA, gives the only Value of BD; which then is not ambiguous, tho' in both Varieties the Perpendicular fell within.

2. When the Perpendicular falls without; that is, when the given Angles are of different Species.

WHEN the obtuſe Value of DA is leſs than BA, BD will be had by ſubtracting either Value of DA from BA; and then BD is ambiguous. But when the obtuſe Value of DA is not leſs than BA, the acute Value of DA, taken from BA, leaves the only Value of BD; which, therefore, is not ambiguous, tho' in both Varieties the Perpendicular fell without.

In the third, we have the ſame Omiſſion; where there are given two Sides BC, CD, *and* B *an Angle oppoſite to* CD *one of them, to find the third Side* BD.

FIRST, we may obſerve, that the Species of DA is always known; for it is of $\left\{\begin{matrix}\text{the ſame}\\ \text{a different}\end{matrix}\right\}$ Affection with the Angle B, when DC is $\left\{\begin{matrix}\text{leſs}\\ \text{greater}\end{matrix}\right\}$ than a Quadrant. And,

If AD be leſs than AB, and alſo the Sum of AD and AB leſs than a Semicircle; then AD, either added to, or ſubtracted from AB, will give the Value of BD; which, therefore, is ambiguous.

But if AD be not leſs than AB, or if their Sum be not leſs than a Semicircle; then their Sum in the former, and their Difference in the latter Variety, ſhall give one ſingle Value of BD; and then is not ambiguous.

The ſeventh Caſe much reſembles the third; for there are given two Sides BC, CD, *and* B *an Angle, oppoſite to* CD *one of them; to find the Angle* BCD, *lying between thoſe two Sides.*

AND here we may obſerve, that the Species of the Angle DCA is known; for it is of $\left\{\begin{matrix}\text{the ſame}\\ \text{a different}\end{matrix}\right\}$ Kind with the Angle B, when DC is $\left\{\begin{matrix}\text{leſs}\\ \text{greater}\end{matrix}\right\}$ than a Quadrant. And,

If DCA be leſs than BCA, and the Sum of DCA and BCA leſs than two Right Angles; then, DCA either added to, or ſubtracted from BCA, will give the Angle BCD; which, therefore, is ambiguous.

If DCA be not leſs than BCA, or the Sum of DCA and BCA not leſs than two right Angles; then their Sum in the former, and their Difference in the latter Variety ſhall give the ſingle Value of BCD; which, then, is not ambiguous.

N. B. If any one will be at the Trouble to make a double Calculation for the Side DC, or the Angle D, as taught in the Remarks on the 9*th* and 10*th* Caſes, they will find the ſeveral Varieties in the 1*ſt*, 3*d*, 5*th*, and 7*th*, to be as here laid down in theſe eaſy Rules.

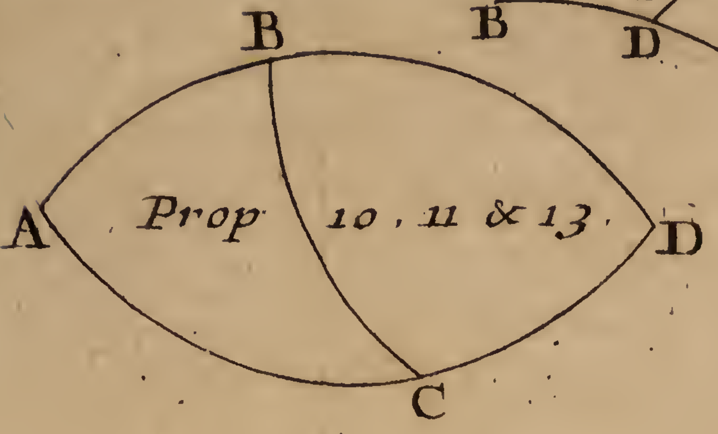
The Truth of theſe Rules may be eaſily deduced from the 10*th*, 13*th*, 18*th*, and 22*d Prop. of this*, and the 2*d*, 8*th*, and 13*th Examples*, following *Prop.* 30. *of this.*

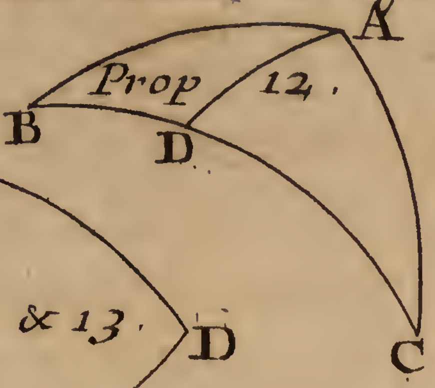
In our third *Caſe* of oblique-plain Triangles, our Author ſhould have added this:

If AB be leſs than BC, the Angle A is ambiguous; otherwiſe not.

Pag. 322.

Spherical Trigonometry


Prop 10, 11 & 13.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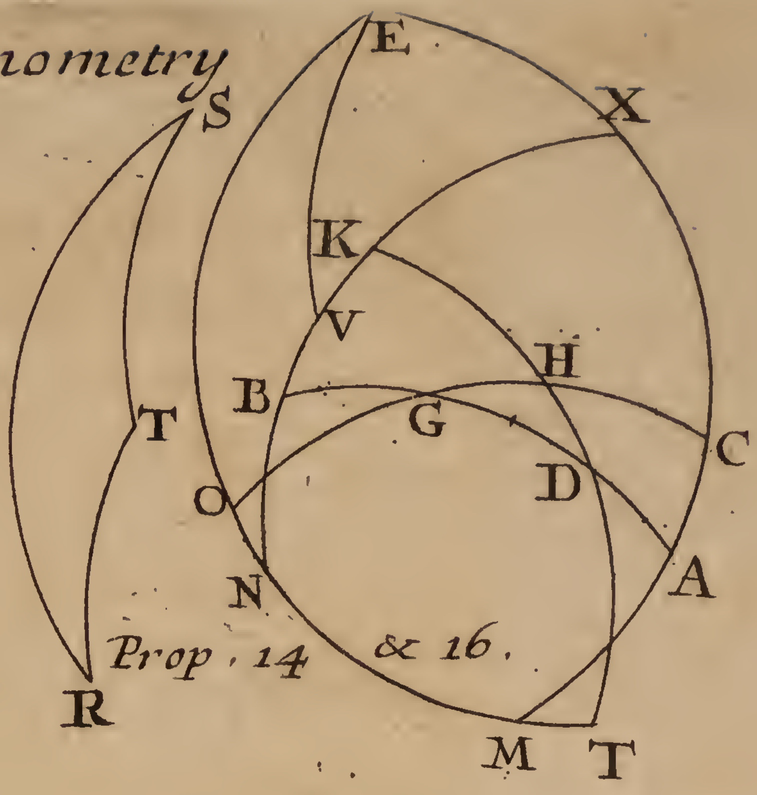
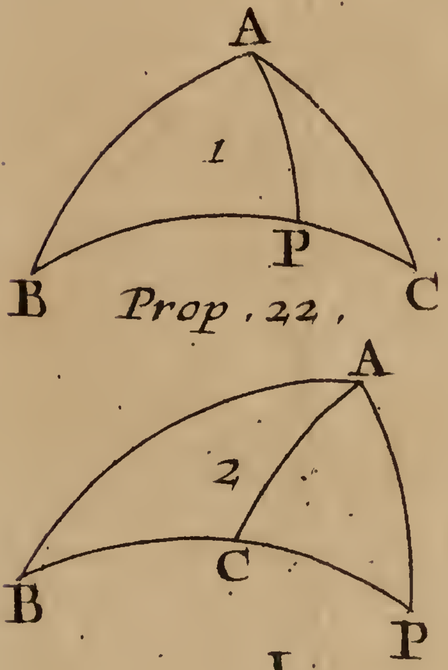

Prop. 14 & 16.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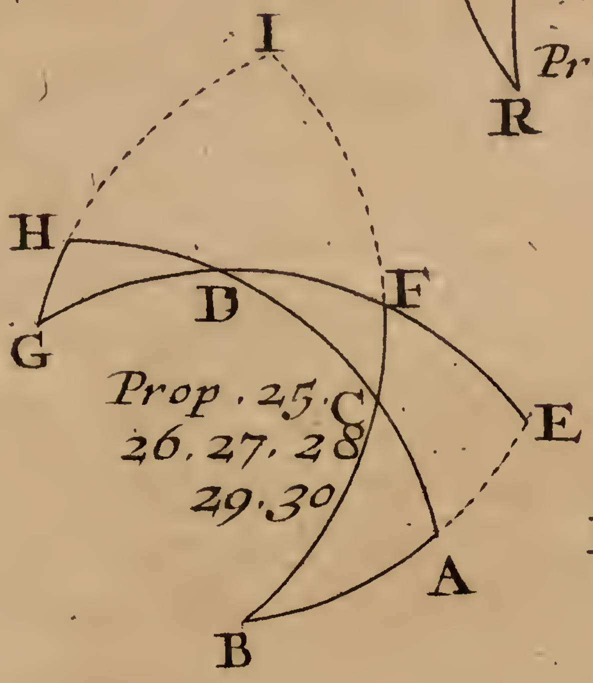

Prop. 25. 26. 27. 28 29. 30

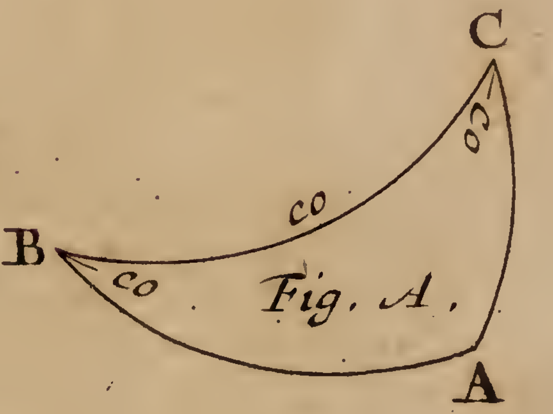


Fig. A.

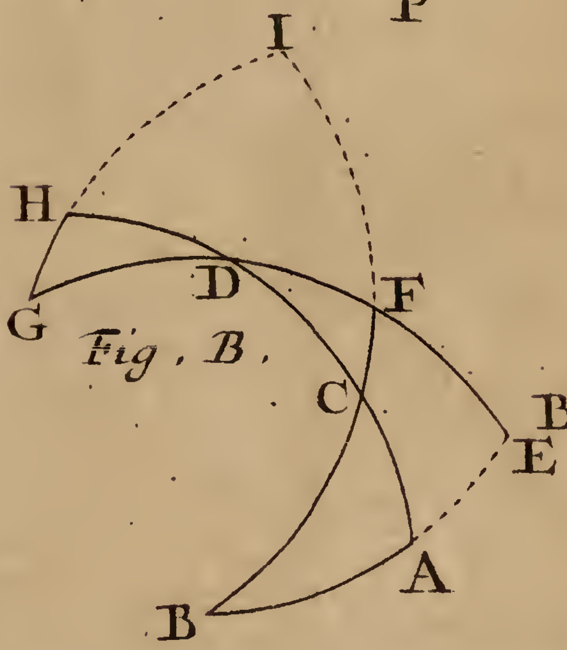
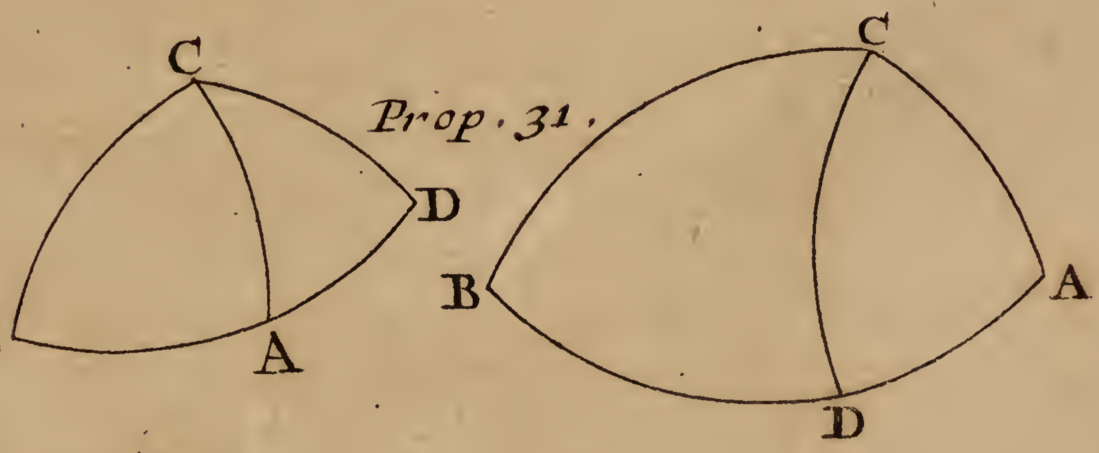


Fig. B.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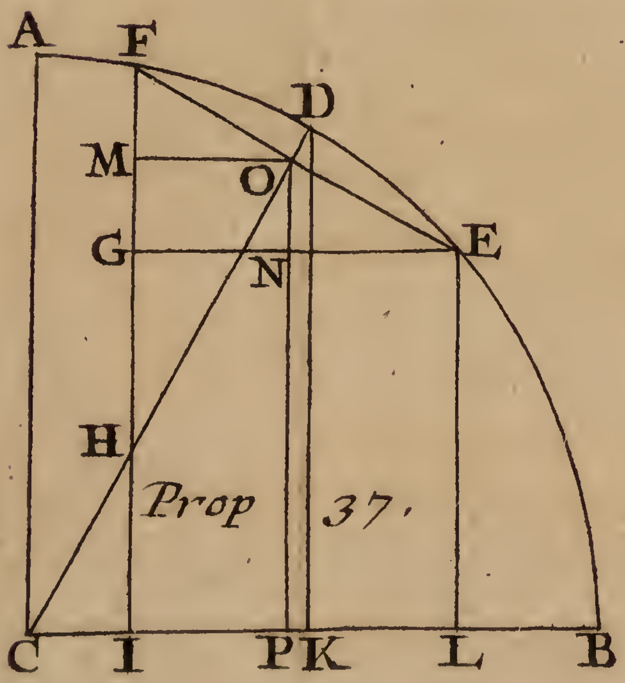
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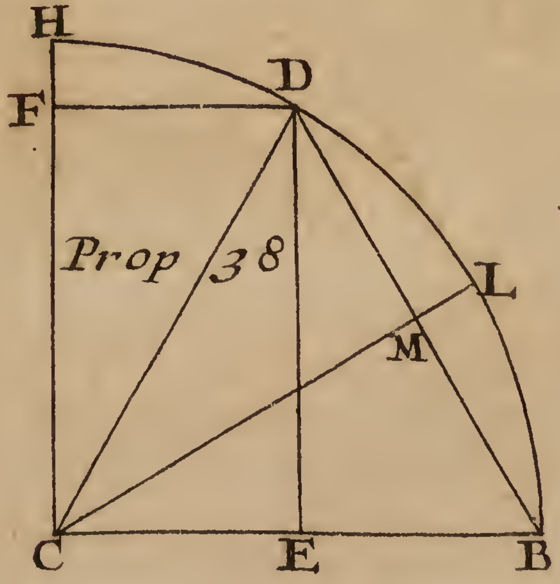
Prop 38

A SHORT
TREATISE
OF THE
Nature *and* Arithmetic
OF
LOGARITHMS.

The PREFACE.

THE Mathematics formerly received conſiderable Advantages; firſt, by the Introduction of the Indian *Characters, and afterwards by the Invention of Decimal Fractions; yet has it ſince reaped at leaſt as much from the Invention of Logarithms, as from both the other two. The Uſe of theſe, every one knows, is of the greateſt Extent, and runs through all Parts of Mathematics. By their Means it is that Numbers almoſt infinite, and ſuch as are otherwiſe impracticable, are managed with Eaſe and Expedition. By their Aſſiſtance the Mariner ſteers his Veſſel, the Geometrician inveſtigates the Nature of the higher Curves, the Aſtronomer determines the Places of the Stars, the Philoſopher accounts for* *other Phænomena of Nature; and laſtly, the Uſurer computes the Intereſt of his Money.*

The Subject of the following Treatiſe has been cultivated by Mathematicians of the firſt Rank; ſome of whom, taking in the whole Doctrine, have indeed written learnedly, but ſcarcely intelligible to any but Maſters. Others, again, accommodating themſelves to the Apprehenſion of Novices, have ſelected out ſome of the moſt eaſy and obvious Properties of Logarithms, but have left their Nature and more intimate Properties untouch'd. My Deſign therefore in the following Tract, is to ſupply what ſeem'd ſtill wanting, viz. *to diſcover and explain the Doctrine of Logarithms, to thoſe who are not yet got beyond the Elements of Algebra and Geometry.*

The wonderful Invention of Logarithms we owe to the Lord Napier, *who was the firſt that conſtructed and publiſhed a Canon thereof, at* Edinburgh, *in the Year* 1614. *This was very graciouſly received by all Mathematicians, who were immediately ſenſible of the extreme Uſefulneſs thereof. And tho' it is uſual to have various Nations contending for the Glory of any notable Invention, yet* Napier *is univerſally allowed the Inventor of Logarithms, and enjoys the whole Honour thereof without any Rival.*

The ſame Lord Napier *afterwards invented another and more commodious Form of Logarithms, which he communicated to Mr.* Henry Briggs, *Profeſſor of Geometry at* Oxford, *who was hereby introduced as a Sharer in the completing thereof: But the Lord* Napier *dying, the whole Buſineſs remaining, was devolved upon Mr.* Briggs, *who, with prodigious Application, and an uncommon Dexterity,*

Dexterity, compoſ'd a Logarithmic Canon, agreeable to that new Form for the firſt twenty Chiliads of Numbers, (or from 1 *to* 20000*) and for eleven other Chiliads,* viz. *from* 90000 *to* 101000. *For all which Numbers he calculated the Logarithms to fourteen Places of Figures. This Canon was publiſhed at* London *in the Year* 1624.

Adrian Vlacq *publiſhed again this Canon at* Gouda *in* Holland, *in the Year* 1628. *with the intermediate Chiliads before omitted, filled up according to* Briggs's *Preſcriptions; but theſe Tables are not ſo uſeful as* Briggs's, *becauſe the Logarithms are continued but to* 10 *Places of Figures.*

Mr. Briggs *alſo has calculated the Logarithms of the Sines and Tangents of every Degree, and the hundredth Parts of Degrees to* 15 *Places of Figures; and has ſubjoined to them the natural Sines, Tangents, and Secants, to* 15 *Places of Figures. The Logarithms of the Sines and Tangents are called artificial Sines and Tangents. Theſe Tables, together with their Conſtruction and Uſe, were publiſh'd after* Briggs's *Death, at* London, *in the Year* 1633. *by* Henry Gillibrand, *and by him called* Trigonometria Britannica.

Since then there have been publiſhed, in ſeveral Places, compendious Tables, wherein the Sines and Tangents, and their Logarithms, conſiſt of but ſeven Places of Figures, and wherein are only the Logarithms of the Numbers from 1 *to* 100000, *which may be ſufficient for moſt Uſes.*

The beſt Diſpoſition of theſe Tables, in my Opinion, is that firſt thought of by Nathanael Roe, *of* Suffolk; *and with ſome Alterations for the* *better, followed by* Sherwin *in his Mathematical Tables, publiſhed at* London *in* 1705; *wherein are the Logarithms from* 1 *to* 101000, *conſiſting of ſeven Places of Figures. To which are ſubjoined the Differences, and proportional Parts, by means of which may be found eaſily the Logarithms of Numbers to* 10000000, *obſerving at the ſame time, that theſe Logarithms conſiſt only of ſeven Places of Figures. Here are alſo the Sines, Tangents, and Secants, with the Logarithms and Differences for every Degree and Minute of the Quadrant, with ſome other Tables of Uſe in practical Mathematics.*

OF THE
Nature *and* Arithmetic
OF
LOGARITHMS.

CHAP. I.

Of the ORIGIN *and* NATURE *of* *LOGARITHMS.*

AS in Geometry, the Magnitudes of Lines are often defined by Numbers; ſo likewiſe, on the other hand, it is ſometimes expedient to expound Numbers by Lines, *viz.* by aſſuming ſome Line which may repreſent Unity, and the Double thereof; the Number 2, the Triple 3, the one half, the Fraction $\frac{1}{4}$, and ſo on. And thus the Geneſis and Properties of ſome certain Numbers are better conceived, and more clearly conſidered, than can be done by abſtract Numbers.

Hence, if any Line a*be drawn into itſelf, the Quantity a^2 produced thereby, is not to be taken as one of two Dimenſions, or as a Geometrical Square, whoſe Side is the Line a, but as a Line that is a third Proportional to ſome Line taken for Unity, and the Line n. So likewiſe, if a^2 be multiplied by a, the Product a^3, will not be a Quantity of three Dimenſions, or a Geometrical Cube, but a Line that is the fourth Term in a Geometrical Progreſſion, whoſe firſt Term is 1, and ſecond a; for the Terms 1, a, a^2, a^3, a^4, a^5, a^6, a^7, *&c.* are in the continual Ratio of 1 to a. And the Indices affixed to the Terms, ſhew the Place or

* *Fig.* 1.

Diſtance that every Term is from Unity. For Example, a^5 is in the fifth Place from Unity, a^6 is the ſixth, or ſix times more diſtant from Unity than a, or a^1, which immediately follows Unity.

If, between the Terms 1 and a, there be put a mean Proportional, which is $\sqrt{a}$, the Index of this will be $\frac{1}{2}$; for its Diſtance from Unity will be one Half of the Diſtance of a from Unity; and ſo $a^{\frac{1}{2}}$ may be written $\sqrt{a}$. And if a mean Proportional be put between a and a^2, the Index thereof will be $\frac{1}{2}$ or $\frac{3}{2}$, for its Diſtance will be ſeſquialteral of the Diſtance a from Unity.

If there be two mean Proportionals put between 1 and a; the firſt of them is the Cube Root of a, whoſe Index muſt be $\frac{1}{3}$, for that Term is diſtant from Unity only by a third Part of the Diſtance of a from Unity; and ſo the Cube Root muſt be expreſſed by $a^{\frac{1}{3}}$. Hence, the Index of Unity is 0; for Unity is not diſtant from itſelf.

The ſame Series of Quantities, geometrically proportional, may be both ways continued, as well deſcending towards the left Hand, as aſcending towards the Right; for the Terms $\frac{1}{a^5}$, $\frac{1}{a^4}$, $\frac{1}{a^3}$, $\frac{1}{a^2}$, $\frac{1}{a}$, 1, a, a^2, a^3, a^4, a^5, &c. are all in the ſame Geometrical Progreſſion. And ſince the Diſtance of a from Unity is towards the right Hand, and poſitive or $+1$, the Diſtance equal to that on the contrary Side, *viz.* the Diſtance of the Term $\frac{1}{a}$ will be Negative or -1, which ſhall be the Index of the Term $\frac{1}{a}$, for which may be written a^{-1}. So likewiſe in the Term a^{-2} the Index -2 ſhews, that that Term ſtands in the ſecond Place from Unity towards the Left Hand, and the Terms a^{-2} and $\frac{1}{a^2}$ are of the ſame Value. Alſo a^{-3} is the ſame as $\frac{1}{a^3}$. For theſe negative Indices ſhew, that the Terms belonging to them, go from Unity the contrary Way to that by which the Terms, whoſe Indices are poſitive, do. Theſe Things premiſed, If on the Line AN, both Ways indefinitely extended, be taken, AC, CE, EG, GI, IL, on the right Hand; and alſo AΓ, ΓΠ, *&c.* on the left, all equal to one another; and if at the Points, Π, Γ, A, C, E, G, I, L, be erected to the Right Line AN, the Perpendiculars ΠΣ, ΓΔ, AB, CD, EF, GH, IK, LM, which let be continually proportional, and repreſent Numbers, whereof AB is Unity: The Lines AC, AE, AG, AI, AL—AΓ,—AΠ, reſpectively expreſs the Diſtances of the Numbers from Unity, or the Place and Order that every Number obtains in the Series of Geometrical Proportionals, according as it is diſtant from Unity. So ſince AG is triple of the Right Line AC, the Number GH ſhall be in the third Place from Unity, if CD be in the firſt: So likewiſe ſhall LM be in the fifth Place, ſince $AL = 5\,AC$. If the Extremities of the Proportionals, Σ, Δ, B, D, F, H, K, M, be joined by Right Lines, the Figure ΣΠ LM will become a Polygon conſiſting of more or leſs Sides, according as there are more or leſs Terms in the Progreſſion.

If the Parts AC, CE, EG, GI, IL, be biſected in the Points, *c*, *e*, *i*, *g*, *l*, and there be again raiſed the Perpendicular *cd*, *ef*, *gh*, *ik*, *lm*, which are mean Proportionals between AB, CD; CD, EF; EF, GH; GH, IK; IK, LM; then there will ariſe a new Series of Proportionals, whoſe Terms, beginning from that which immediately follows Unity, are double of thoſe in the firſt Series, and the Difference of the Terms are become leſs, and approach nearer to a Ratio of Equality than before. Likewiſe in this new Series, the Right Lines AL, AC, expreſs the Diſtances of the Terms LM, CD, from Unity; *viz.* Since AL is ten times greater than A*c*, LM ſhall be the tenth Term of the Series from Unity: And becauſe A*e* is three times greater than A*c*, *ef* will be the third Term of the Series, if *cd* be the firſt; and there ſhall be two mean Proportionals between AB and *ef*; and between AB, and LM, there will be nine mean Proportionals.

And if the Extremities of the Lines B*d*, D*f*, F*h*, H, *&c.* be joined by Right Lines, there will be a new Polygon made, conſiſting of more, but ſhorter Sides than the laſt.

If, again, the Diſtances A *c*, *c* C, C *e*, *e* E, *&c.* be ſuppoſed to be biſected, and mean Proportionals between every two of the Terms, be conceived to be put at thoſe middle Diſtances; then there will ariſe another Series of Proportionals, containing double the Number of Terms from Unity than the former does; but the Differences of the Terms will be leſs, and if the Extremities of the Terms be joined, the Number of the Sides of the Polygon will be augmented according to the Number of Terms; and the Sides thereof will be leſſer, becauſe of the Diminution of the Diſtances of the Terms from each other.

Now, in this new Series, the Diſtances A L, A C, *&c.* will determine the Orders or Places of the Terms; *viz.* If A L be five times greater than A C, and C D be the fourth Term of the Series from Unity, then L M will be the twentieth Term from Unity.

If in this manner mean Proportionals be continually placed between every two Terms, the Number of Terms at laſt will be made ſo great, as alſo the Number of the Sides of the Polygon, as to be greater than any given Number, or to be infinite; and every Side of the Polygon ſo leſſened, as to become leſs than any given Right Line; and conſequently the Polygon will be changed into a curve-lined Figure; for any curve-lin'd Figure may be conceiv'd as a Polygon, whoſe Sides are infinitely ſmall, and infinite in Number.

A Curve deſcribed after this manner, is called *Logarithmical*; in which, if Numbers be repreſented by Right Lines ſtanding at Right Angles to the Axis A N, the Portion of the Axis intercepted between any Number and Unity, ſhews the Place or Order that that Number obtains in the Series of Geometrical Proportionals, diſtant from each other by equal Intervals. For Example, if A L be five times greater than A C, and there are a thouſand Terms in continual Proportion from Unity to L M; then will there be two hundred Terms of the ſame Series from Unity to C D, or C D ſhall be the two hundredth Term of the Series from Unity; and let the Number of Terms from A B to L M be ſuppoſed what it will, then the Number of Terms from A B to C D, will be one fifth Part of that Number.

The Logarithmical Curve may alſo be conceived to be deſcribed by two Motions, one of which is equable, and the other accelerated, or retarded, according to a given Ratio. For Example, if the Right Line AB moves uniformly along the Line AN, ſo that the End thereof deſcribes equal Spaces in equal Times; and, in the mean time, the ſaid Line AB ſo increaſes, that the Increments thereof, generated in equal Times, be proportional to the whole increaſing Line; that is, if AB, in going forward to *cd*, be increaſed by the Increment *od*, and in an equal Time when it is come to CD, the Increment thereof is D*p*, and D*p* to *dc* is as *do* is to AB, that is, if the Increments generated in equal times are always proportional to the Wholes; or, if the Line AB, moving the contrary Way, diminiſhes in a conſtant Ratio, ſo that while it goes thro' the equal Spaces, the Decrements AB—ΓΔ ΓΔ, —ΠΣ, are Proportionals to AB, ΓΔ; then the End of the Line, increaſing or decreaſing in the ſaid manner, deſcribes the Logarithmical Curve: For ſince AB : *do* :: *dc* : D*p* :: DC : *fq*, it ſhall be (by Compoſition of Ratio) as AB : *dc* :: *dc* : DC :: DC : *fe*, and ſo on.

By theſe two Motions, *viz.* the one equable, and the other proportionally accelerated or retarded, the Lord *Napier* laid down the Origin of Logarithms, and called the Logarithm of the Sine of any Arc, *That Number which neareſt defines a Line that equally increaſes, while, in the mean time, the Line expreſſing the whole Sine proportionally decreaſes to that Sine.*

It is manifeſt from this Deſcription of the Logarithmic Curve, that all Numbers at equal Diſtances are continually proportional. It is alſo plain, that if there be four Numbers AB, CD, IK, LM, ſuch, that the Diſtance between the firſt and ſecond be equal to the Diſtance between the third and the fourth: Let the Diſtance from the ſecond to the third be what it will, theſe Numbers will be proportional. For, becauſe the Diſtances AC, IL, are equal, AB ſhall be to the Increment D*s*, as IK is to the Increment MT. Wherefore (by Compoſition) AB : DC :: IK : ML. And contrariwiſe, if four Numbers be proportional, the Diſtance between the firſt and the ſecond ſhall be equal to the Diſtance between the third and the fourth.

The Diſtance between any two Numbers is called the Logarithm of the Ratio of thoſe Numbers, and indeed doth not meaſure the Ratio itſelf, but the Number of Terms in a given Series of Geometrical Proportionals proceeding from one Number to another, and defines the Number of equal Ratio's by the Compoſition whereof the Ratio's of Numbers are known.

If the Diſtance between any two Numbers be double to the Diſtance between two other Numbers, then the Ratio of the two former Numbers ſhall be the Duplicate of the Ratio of the two latter. For let the Diſtance IL between the Numbers IK, LM, be double to the Diſtance A*c*, between the Numbers AB, *cd*; and ſince IL is biſected in *l*, we have $Ac = Il = lL$; and the Ratio of IK to *lm* is equal to the Ratio of AB to *cd*; and ſo the Ratio of IK to LM, the Duplicate of the Ratio of IK to *lm*, (*by Def.* 10. *El.* 5.) ſhall be the Duplicate of the Ratio of AB to *cd*.

In like manner, if the Diſtance EL be triple of the Diſtance AC, then will the Ratio of EF to LM, be triplicate of the Ratio of AB to CD: For, becauſe the Diſtance is triple, there ſhall be three times more Proportionals from EF to LM, than there are Terms of the ſame Ratio from AB to CD; and the Ratio of EF to LM, as alſo of AB to CD, is compounded of the equal intermediate Ratio's (*by Def.* 5. *El.* 6.). And ſo the Ratio of EF to LM, compounded of three times a greater Number of Ratio's, ſhall be triplicate of the Ratio of AB to CD. So likewiſe, if the Diſtance GL be quadruple of the Diſtance A*c*, then ſhall the Ratio of GH to LM, be quadruplicate of the Ratio of AB to *cd*.

The Logarithm of any Number is the Logarithm of the Ratio of Unity to that Number, or it is the Diſtance between Unity and that Number. And ſo Logarithms expreſs the Power, Place, or Order which every Number, in a Series of Geometrical Progreſſionals, obtains from Unity. For Example, if there be 10000000 proportional Numbers from Unity to the Number 10, that is, if the Number 10 be in the 100000000th Place from Unity; then it will be found, by Computation, that in the ſame Series from Unity, to 2, there are 3010300 proportional Terms; that is, the Number 2 will ſtand in the 3010300th Place. In like manner, from Unity to 3, there will be found 4771213, proportional Terms, which Number defines the Place of the Number 3. The Numbers 10000000, 3010300, 4771213, ſhall be the Logarithms of the Numbers 10, 2, and 3.

If the firſt Term of the Series from Unity be called y, the ſecond Term will be y^2, the third y^3, *&c.* And ſince the Number 10 is the 10,000,000th Term of the Series, then will $y^{10000000}=10$. Alſo $y^{3010300}=2$. Alſo $y^{4771213}=3$; and ſo on.

Wherefore all Numbers ſhall be ſome Powers of that Number which is the firſt from Unity; and the Indices of the Powers are the Logarithms of the Numbers.

Since Logarithms are the Diſtances of Numbers from Unity, as has been ſhewn, the Logarithm of Unity ſhall be 0; for Unity is not diſtant from itſelf, but the Logarithms of Fractions are negative, or deſcending below nothing; for they go on the contrary Way. And ſo if Numbers increaſing proportionally from Unity, have poſitive Logarithms, or ſuch as are affected with the Sine $+$; then Fractions or Numbers in like manner decreaſing, will have negative Logarithms, or ſuch as are affected with the Sign $-$; which is true when Logarithms are conſidered as the Diſtances of Numbers from Unity.

But if Logarithms take their Beginning not from an integral Unit, but from a Unit that is in ſome Place of decimal Fractions; for Example, from the Fraction $\frac{1}{10000000000}$; then all Fractions greater than this, will have poſitive Logarithms; and thoſe that are leſs, will have negative Logarithms. But more ſhall be ſaid of this hereafter.

Since in the Numbers continually proportional, DC, EF, GH, IK, *&c.* the Diſtances CE, EG, GI, *&c.* are equal, the Logarithms AC, AE, AG, AI, *&c.* of thoſe Numbers, ſhall be equidifferent, or the Differences of them ſhall be equal: And ſo the Logarithms of proportional Numbers are all in an Arithmetical Progreſſion; and from hence proceeds that common Definition of Logarithms, that Logarithms are Numbers which, being adjoined to Proportions, have equal Differences.

In the firſt Kind of Logarithms that *Napier* publiſhed, the firſt Term of the continual Proportionals was placed only ſo far diſtant from Unity, as that Term exceeded Unity. For Example, if *v n* be the firſt Term of the Series from Unity A B, the Logarithm thereof, or the Diſtance A *n*, or B *y*, was, according to him, equal to *v y*, or the Increment of the Number above Unity. As ſuppoſe *v n* be 1,0000001, he placed 0,0000001 for its Logarithm *A n*; and from hence, by Computation, the Number 10 ſhall be the 23025850th Term of the Series, which Number therefore is the Logarithm of 10 in this Form of Logarithms, and expreſſes its Diſtance from Unity in ſuch Parts whereof *v y* or A *n* is one.

But this Poſition is intirely at Pleaſure; for the Diſtance of the firſt Term may have any given Ratio to the Exceſs thereof above Unity, and according to that various Ratio, (which may be ſuppoſed at Pleaſure) that is between *v y* and B *y*, the Increment of the firſt Term above Unity, and the Diſtance of the ſame from Unity, there will be produced different Forms of Logarithms.

This firſt Kind of Logarithms was afterwards changed by *Napier*, into another more convenient one, wherein he put the Number 10 not as the 23025850th Term of the Series, but the 1000,0000th; and in this Form of Logarithms, the firſt Increment *v y* ſhall be to the Diſtance B *y*, or A *n*, as Unity, or A B, is to the Decimal Fraction 0,4342994, which therefore expreſſes the Length of the Subtangent A T. *Fig.* 4.

After *Napier*'s Death, the excellent Mr. *Henry Briggs*, by great Pains, made and publiſhed Tables of Logarithms according to this Form. Now ſince in theſe Tables the Logarithm of 10, or the Diſtance thereof from Unity, is 1,0000000, and 1, 10, 100, 1000, 10000, *&c.* are continual Proportionals, they ſhall be equidiſtant. Wherefore the Logarithm of the Number 100 ſhall be 2,0000000; of 1000, 3,0000000; and the Logarithm of 10000 ſhall be 4,0000000; and ſo on.

Hence the Logarithms of all Numbers between 1 and 10, muſt begin with 0, or 0 muſt ſtand in the firſt Place to the left Hand; for they are leſſer than the Logarithm of the Number 10, whoſe Beginning is Unity; and the Logarithms of the Numbers between 10 and 100 begin with Unity; for they are greater than 1,0000000, and leſs than 2,0000000. Alſo the Logarithms between 100 and 1000, begin with 2; for they are greater than the Logarithm of 100, which begins with 2, and leſs than the Logarithm of 1000, that begins with 3. In the ſame manner it is demonſtrated, that the firſt Figure to the left Hand of the Logarithms between 1000 and 10000, muſt be 3; and the firſt Figure to the left Hand of the Logarithms between 10000 and 100000, will be 4; and ſo on.

The firſt Figure of every Logarithm to the left Hand, is called the Characteriſtic or Index, becauſe it ſhews the higheſt or moſt remote Place of the Number from the Place of Units. For Example, if the Index of a Logarithm be 1, then the higheſt or moſt remote Place from Unity of the correſpondent Number to the left Hand, will be the Place of Tens. If the Index be 2, the moſt remote Figure of the correſpondent Number ſhall be in the ſecond Place from Unity; that is, it ſhall be in the Place of Hundredths; and if the Index of a Logarithm be 3, the laſt Figure of the Number anſwering to it, ſhall be in the Place of Thouſandths. The Logarithms of all Numbers that are in decuple or ſubdecuple Progreſſion, only differ in their Characteriſtics, or Indices, they being written in all other Places with the ſame Figures. For Example, the Logarithms of the Numbers 17, 170, 1700, 17000, are the ſame, unleſs in their Indices; for ſince 1 is to 17, as 10 to 170, and as 100 to 1700, and as 1000 to 17000; therefore the Diſtances between 1 and 17, between 10 and 170, between 100 and 1700, and between 1000 and 17000, ſhall be all equal. And ſo, ſince the Diſtance between 1 and 17, or the Logarithm of the Number 17, is 1.2304489, the Logarithm of the Number 170 will be=2.2304489, and the Logarithm of the Number 1700 ſhall be 3.2304489, becauſe the Logarithm of the Number 100=2.0000000. In like manner, ſince the Logarithm of the Number 1000=3.0000000, the Logarithm of the Number 17000 ſhall 4.2304489.

So alſo the Numbers, 6748. 674, 8. 67, 48. 6, 748. 0, 6748. 0, 06748, are continual Proportionals in the Ratio of 10 to 1; and ſo their Diſtances from each other ſhall be equal to the Diſtance or Logarithm of the Number 10, or equal to 1,0000000. And ſo, ſince the Logarithm of the Number 6748 is 3,8291751, the Logarithms of the other Numbers ſhall be as in the Margin; where you may obſerve, that the Indices of the laſt two Logarithms are only negative, and the other Figures poſitive; and ſo, when thoſe other Figures are to be added, the Indices muſt be ſubtracted, and contrariwiſe.

6748	3,8291751
674,8	2,8291751
67,48	1,8291751
6,748	0,8291751
0,6748	−1,8292751
0,06748	−2,8291751

CHAP. II.

Of the Arithmetic of Logarithms in whole Numbers, or whole Numbers adjoined to Decimal Fractions. Fig. 2.

BEcauſe, in Multiplication, Unity is to the Multiplier, as the Multiplicand is to the Product, the Diſtance between Unity and the Multiplier, ſhall be equal to the Diſtance between the Multiplicand and the Product. If therefore the Number G H be to be multiplied by the Number E F, the Diſtance between G H and the Product muſt be equal to the Diſtance A E, or to the Logarithm of the Multiplier; and ſo, if G L be taken equal to A E, the Number L M ſhall be the Product; that is, if the Logarithm of the Multiplicand A G be added to the Logarithm of the Multiplier A E, the Sum ſhall be the Logarithm of the Product.

In Diviſion, the Diviſor is to Unity, as the Dividend is to the Quotient; and ſo the Diſtance between the Diviſor and Unity ſhall be equal to the Diſtance between the Dividend and the Quotient. So if L M be to be divided by E F, the Diſtance E A ſhall be equal to the Diſtance between L M and the Quotient;

and ſo, if L G be taken equal to E A, the Quotient will be at G; that is, if from A L, the Logarithm of the Dividend, be taken G L, or A E, the Logarithm of the Diviſor, there will remain A G, the Logarithm of the Quotient.

And from hence it appears, that whatſoever Operations in common Arithmetic are performed by multiplying or dividing of great Numbers, may be much eaſier, and more expediently, done by the Addition or Subtraction of Logarithms.

For Example, Let the Number 7589 be to be multiplied by 6757. Now if the Logarithms of thoſe Numbers be added together, as in the Margin, their Sum will be the Logarithm of the Product, whoſe Index 7 ſhews, that there are ſeven Places of Figures, beſides Unity, in the Product; and in ſeeking this Logarithm in Tables, or the neareſt equal to it, I find that the Number anſwering thereto, which is leſſer than the Product, is 51278000; and the Number greater than the Product is 51279000; and if the adjoined Differences, and proportional Parts, be taken, the Numbers that muſt be added to the Place of Hundreds and Tens in the Product are 87; and that which muſt be added in the Place of Unity, will neceſſarily be 3, ſince ſeven times 9=63; and ſo the true Product ſhall be 51278873. If the Index of the Logarithm had been 8 or 9, then the Numbers to be added in the Place of Hundredths or Tenths could not be had from thoſe Tables of Logarithms which conſiſt but of 7 Places of Figures, beſides the Characteriſtic; and ſo in this Caſe the *Vlaquian* or *Briggian* Tables ſhould be uſed; in the former of which, the Logarithms are all to ten Places of Figures, and in the latter to fourteen.

Log. 3. 8801846
Log. 3. 8297539
Log. 7. 7099385

If the Number 78956 be to be divided by 278, by ſubtracting the Logarithm of the Diviſor from the Logarithm of the Dividend, the Logarithm of the Quotient will be had. And to this Logarithm, the Number 282, 719 anſwers; which therefore ſhall be the Quotient.

Log. 4. 8954004
Log. 2. 4440448
Log. 2. 4513556

Becauſe Unity, any aſſumed Number, the Square thereof, the Cube, the Biquadrate, *&c.* are all continual Proportionals, their Diſtances from each other ſhall be equal to one another. And ſo it is manifeſt, that the Diſtance of the Square from Unity, is double of the Diſtance of its Root from the ſame: Alſo the Diſtance of the Cube is triple of the Diſtance of its Root; and the Diſtance of the Biquadrate, is quadruple of the Diſtance of its Root from Unity, *&c.* And ſo, if the Logarithm of any Number be doubled, we ſhall have the Logarithm of its Square; if it be tripled, we ſhall have the Logarithm of its Cube; and if it be quadrupled, the Logarithm of its Biquadrate. And contrariwiſe, if the Logarithm of any Number be biſected, we ſhall have the Logarithm of the ſquare Root thereof: Moreover, a third Part of the ſaid Logarithm will be the Logarithm of the Cube Root of the Number; and a fourth Part, the Logarithm of the Biquadrate Root of that Number.

Hence, the Extractions of all Roots are eaſily performed, by dividing a Logarithm into as many Parts as there are Units in the Index of the Power. So if you want the Square Root of 5, the half of 0,6989700 muſt be taken, and then that half 0.3494850 will be the Logarithm of the Square Root of 5, or the Logarithm of $\sqrt{5}$, to which the Number 2.23606 nearly anſwers.

CHAP. III.

Of the Arithmetic of Logarithms, when the Numbers are Fractions. Fig. 3.

WHEN Fractions are to be worked by Logarithms, it is neceſſary, for avoiding the Trouble of adding one Part of a Logarithm, and ſubtracting the other, that Logarithms do not begin from an integral Unit, but from ſome Unit that is the tenth or hundredth Place of Decimal Fractions: For Example, let P O be $\frac{1}{10000000000}$, and from this let the Logarithm begin. Now this Fraction is ten times more diſtant from Unity to the left Hand, than the Number 10 is diſtant therefrom to the right; for there are 10 proportional Terms in the Ratio of 10 to 1, from Unity to P O. And ſo, if A B be Unity, the Logarithm thereof, according to this Suppoſition, will not be 0, but O A will be=10.0000000; for the Diſtance of any Tenth from Unity is 1.0000000, whence the Diſtance of the Number 10 from P O will be 11.0000000. Alſo the Diſtance of the Number 100 from P O, or its Logarithm, beginning from P O, ſhall be 12.0000000; and the Logarithm of 1000, or the Diſtance from P O, will be 13.0000000. And thus, the Indices of all Logarithms are augmented by the Number 10; and thoſe Fractions whoſe Indices are —1, or —2, or —3, *&c.* are now made 9, 8, or 7, *&c.*

But if Logarithms begin from the Place of a Fraction, whoſe Numerator is Unity, and Denominator Unity with 100 Cyphers added to it, (which they muſt do when Fractions occur that are leſs than P O) then that Fraction will be 100 times more diſtant from Unity, than 10 is diſtant from it; and ſo the Logarithm of Unity will have 100 for the Index thereof. And the Logarithm of any Tens will have 101 for the Index, that of any Hundreds 102, and ſo on; all the Indices being augmented by the Number 100.

The Logarithms of all Fractions that are greater than P O (whereat they begin) will be poſitive. And ſince the Numbers 10, 1, $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, *&c.* are in a continued Geometrical Progreſſion, they will be equally diſtant from each other; and accordingly their Logarithms will be equidifferent: And ſo, when the Logarithm of 10 is 11.0000000, and the Logarithm of Unity is 10.0000000, and the Logarithm of the Fraction $\frac{1}{10}$ will be 9.0000000, and the Logarithm of the Fraction $\frac{1}{100}$ will be 8.0000000; and in like manner, the Index of the Logarithm of $\frac{1}{1000}$ will be 7. Alſo for the ſame Reaſon, if the Index of the Logarithm of Unity be 100, and of 10 be 101, then will the Index of the Logarithm of the Fraction $\frac{1}{10}$ be 99, and the Index of the Logarithm of $\frac{1}{100}$ will be 98, and the Index of the Logarithm of the Fraction $\frac{1}{1000}$ ſhall be 97, *&c.* And theſe Indices ſhew in what Place from Unity, the firſt Figure of the Fraction, not being a Cypher, muſt be put. For Example, if the Index be 4, the Diſtance thereof from the Index of Unity, (which is 10) *viz.* 6, ſhews that the firſt Significative Figure of the Decimal, is in the ſixth Place from Unity; and therefore five Cyphers are to be prefixed thereto towards the left Hand. So alſo, if the Index of Unity be 100, and the Index of the Fraction be 80, the firſt Figure thereof ſhall be in the 20th Place from Unity, and 19 Cyphers are to be prefixed thereto.

Now, let it be required to multiply the Fraction GH by the Fraction DC. Becauſe Unity is to the Multiplier, as the Multiplicand is to the Product; the Diſtance between Unity and the Multiplier ſhall be equal to the Diſtance between the Multiplicand and the Product. Therefore, if there be taken $GI = AC$, the Product IK ſhall be at I. And accordingly, if from OG, the Logarithm of the Multiplicand, there be taken GI or AC, there will remain OI, the Logarithm of the Product. But $AC = OA - OC$, which taken from OG, there will remain $OG + OC - OA = OI$; that is, if the Logarithm of the Multiplier and Multiplicand be added together, and from the Sum be taken the Logarithm of Unity, (which is always expreſſed by 10 or 100 with Cyphers) the Logarithm of the Product will be had. For Example, let the Decimal Fraction 0,00734 be to be multiplied by the Fraction 0,000876. Set down 100 for the Index of the Logarithm of Unity, and then the Logarithms of the Fractions will be as in the Margin; which being added together, and the Logarithm of Unity being taken away from the Sum, the Remainder is the Logarithm of the Product, whoſe Index 94 ſhews, that the firſt Figure of the Product is in the ſixth Place from Unity; and ſo there muſt be five Cyphers prefixed, and then the Product will be, 00000642984.

$$\begin{array}{r} 97,8656961 \\ 96,9425041 \\ \hline 94.8082002 \end{array}$$

In Diviſion, the Diviſor is to Unity, as the Dividend is to the Quotient; and ſo the Diſtance between the Diviſor and Unity ſhall be equal to the Diſtance between the Dividend and the Quotient. And ſo, if the Fraction IK be to be divided by DC, you muſt take $IG = CA$, and the Place of the Quotient ſhall be G. But $CA = OA - OC$, which being added to OI, we have $OA + OI - OC = OG$; that is, if the Logarithm of Unity be added to the Logarithm of the Dividend, and from the Sum be taken the Logarithm of the Diviſor, there will remain the Logarithm of the Quotient; ſo if the Number CD be to be divided by I K, you muſt take the Diſtance C S = I A, and then S T will be the Quotient, whoſe Logarithm is O A + O C — O I. Let C D = 0.347, I K = 0.00478. Then add the Logarithm of Unity to the Logarithm of C D; that is, put 1 or 10 before the Index thereof, and from that ſubtract the Logarithm of the Diviſor, and the Remainder will be the Logarithm of the Quotient, whoſe Index 11. ſhews, that the Quotient is between the Numbers 10 and 100; and I ſeek the Number anſwering the Logarithm, which I find to be 72, 542. If the Logarithm of a Vulgar Fraction, for Example, $\frac{7}{8}$, be required, the Logarithm of Unity muſt be added to the Logarithm of the Numerator 7; or, which is all one, you muſt put 10 or 100 before the Index thereof, and ſubduct from it the Logarithm of the Denominator 8, and there will remain the Logarithm of the Vulgar Fraction $\frac{7}{8}$, or the Decimal .875.

$$\begin{array}{r} 19.5403295 \\ \underline{7.6794279} \\ 11.8609016 \end{array}$$

$$\begin{array}{r} 10.8450980 \\ \underline{0.9030900} \\ 9.9420080 \end{array}$$

If the Powers of any Fraction D C be required, you muſt aſſume E C, E G, G I, I L, each equal to A C; and then E F will be the Square, G H the Cube, and I K the Biquadrate of the Number D C; for they are continually proportional from Unity. Beſides, A E = 2 A C = 2 A O — 2 O C; whence O E = O A — A E = 2 O C — O A; that is, the Logarithm of the Square is the Double of the Logarithm of the Root, leſs the Logarithm of Unity. In like manner, ſince A G = 3 A C = 3 O A — 3 O C, we ſhall have O G = O A — A G = 3 O C — 2 O A = the Logarithm of the Cube = triple the Logarithm of the Root, — the Double of the Logarithm of Unity. For the ſame Reaſon, becauſe A I = 4 A C = 4 O A — 4 O C, we have O I = 4 O C — 3 O A, which is the Logarithm of the Biquadrate. And univerſally, if the Power of a Fraction be n, and the Logarithm L, then ſhall the Logarithm of the Power $n = n\text{L} - n\text{OA} + \text{OA}$; that is, if the Logarithm of a Fraction be multiplied by n, and from the Product be taken the Logarithm of Unity, multiplied by $n - 1$, the Logarithm of the Power n of that Fraction will be had.

For Example, if it is required to find the 6th Power of the Fraction $\frac{1}{20}=,05$ the Logarithm of this Fraction is 8.6989700, which being multiplied by 6, gives the Number 52. 1938200; and if from 52 the Number 50, which is the Index of the Logarithm of Unity drawn into 5, be taken away, the Remainder will be the Logarithm of the 6th Power, *viz.* 2.1938200, to which the Number ,0000000 15625 anſwers. For the Index 2 ſhews, that 7 Cyphers muſt be put before the firſt Figure.

If the 8th Power of the Fraction ,05 be required, by multiplying the Logarithm by 8, there will be produced 69.5917600; and ſince 70, which is ſeven times the Index of the Logarithm of Unity, cannot be taken from 69, unleſs we run into negative Numbers, the Index of the Logarithm of Unity muſt be ſuppoſed 100, and then the Index of the Logarithm of the Fraction will be 98. Now this Logarithm, drawn into 8, gives 789. 5917600; and if 700, which is 7 times the Index of the Logarithm of Unity, be taken from 789, there will remain 89. 5917600, the Logarithm of the 8th Power of the Fraction $\frac{1}{20}$, whoſe correſpondent Number is ,0000000000 39062: For ſince the Index is 89, and the Difference thereof from 100 is 11; the firſt ſignificative Figure of the Fraction ſhall be in the 11th Place from Unity; and ſo there muſt be 10 Cyphers placed before it.

If the Roots of the Powers of Fractions be deſired, for Example, the Square Root of the Fraction E F, becauſe the Root is a mean Proportional between the Fraction and Unity, you muſt biſect A E in C, and then C D will be the Square Root of the Fraction E F. But $AC = \frac{1}{2} AE = \frac{OA - OE}{2}$; and ſo the Logarithm of the Root $= OA - AC = \frac{OA + OE}{2}$. And if the Cube Root of the Fraction G H be ſought, this ſhall be the firſt of two mean Proportionals between Unity and G H; and ſo, if A G be divided into three equal Parts, the firſt of which is A C, then C D ſhall be the Root ſought, and becauſe $AC = \frac{1}{3} AG = \frac{OA - OG}{3}$, if this be taken from O A, there will remain $\frac{2OA+OG}{3}=OC=$Logarithm of the Cube Root of the Fraction GH. So likewiſe the biquadrate Root of the Fraction IK will be had, by dividing AI into four equal Parts; for the Root is the firſt of three mean Proportionals between Unity and the Fraction; and conſequently, if $AC=\frac{1}{4}AI$, then will CD be the biquadrate Root of the Fraction IK. But $\frac{1}{4}AI=\frac{OA-OI}{4}$; and ſo $OC=OA-AC=\frac{3OA+OI}{4}$.

And univerſally, if the Root of any Power n of the Fraction LM be required, the Logarithm of the Root thereof will be $\frac{nOA-OA+OL}{n}$; that is, if the Number $n-1$ be prefix'd to the Index of the Logarithm, and the Logarithm thus augmented be divided by n, the Quotient will give the Logarithm of the Root ſought: So if the Cube Root of the Fraction $\frac{1}{2}$ or .5 be ſought, you muſt place $2=n-1$ (ſince the Cube Root is required) before the Logarithm thereof, and there will be had 29,6989700, a third Part of which is 9,8996566, which is equal to the Logarithm of the Cube Root of the Fraction $\frac{1}{2}$, and the Number ,7937, anſwering to this Logarithm, is the Root ſought.

CHAP. IV.

Of the Rule of Proportion by Logarithms.

THE Rule of Proportion ſhews how, by having three Numbers given, a fourth Proportional to them may be found; *viz.* if the ſecond and third Terms be multiplied by one another, and the Product divided by the firſt Term, then will the Quotient be the fourth proportional Term ſought. But this fourth Term is much eaſier found by Logarithms; for if the Logarithm of the firſt Term be taken from the Sum of the Logarithms of the ſecond and third Term, the Number remaining will be the Logarithm of the fourth ſought.

Or this may be done ſomething eaſier yet, if inſtead of the Logarithm of the firſt Term be taken its Complement Arithmetical, or the Difference of that Logarithm, and the Number 10. 0000000, which is done by ſetting down the Difference between each Figure of the Logarithm, and the Figure 9; for then, if that Arithmetical Complement be added to the Sum of the other two Logarithms, and if Unity, which is the firſt Figure to the left Hand, be taken from the Sum, the Remainder will be the Logarithm of the fourth Term ſought; and ſo by this Way, Logarithms of the fourth Term are found by only one Addition of three Numbers. The Reaſon of this will be manifeſt from hence: Let there be three Numbers A, B, C, from which the firſt is to be taken from the Sum of the ſecond and third. Now this may not only be done by the common Way, but likewiſe, if there be any other third Number E taken, and from this there be taken A, there will remain E—A, and if the Numbers B, C, and E—A be all added together, and from their Sum be taken E, there will remain B+C—A. So, if the Number 15 be to be taken from 23, take the Complement of the Number 15 to 100, which is 85, and add this Number to 23, and the Sum will be 108, from which 100 being taken, there remains the Number 8.

$$\begin{array}{r} 85 \\ \underline{23} \\ 108 \end{array}$$

Hence follow ſome Trigonometrical Examples of the Rule of Proportion ſolv'd by Logarithms.

Let ABC be a Right-angled Triangle, wherein are given, the Angle A 36 Degrees 46′, the Angle B 98 Degrees 32′, and the Side BC 3478, the Side AC is required. Say (*by Caſe* 1. *of plain Trig.*) as the Sine of the Angle A is to the Sine of the Angle B, ſo is BC to AC. And becauſe the Logarithm Sine of the Angle A is the firſt Term of the Analogy, I ſubſtitute its Complement Arithmetical for the ſame, and add the Logarithm of BC, the Logarithm of S, B, and the ſaid Complement, all three together, and rejećt Unity,

Arith. Comp. S, A.	0.2228938
Log. Sin. B.	9.9951656
Log. BC.	3.5413296
Log. AC.	×3.7593890

Unity, which is in the firſt Place to the left Hand; and then the Logarithm of the Side A C will be given, and the Number anſwering thereto is 5706,306 equal to the Side ſought A C.

Let there be a ſpherical Triangle A B C, in which are given all the Sides, *viz.* B C = 30 Degrees, A B = 24 Degrees 4′, and A C = 42 Degrees 8′, the Angle B is required. Let B A be produced to M, ſo that B M = B C; then will A M, the Difference of the Sides B C, B A, be equal to 5 Degrees 56′. Now (*by Caſe* 11. in oblique-angled ſpherical Triangles) ſay, As the Rectangle under the Sines of the Legs, is to the Square of Radius, ſo is the Rectangle under the Sines of the Arcs $\frac{AC+AM}{2}$, $\frac{AC-AM}{2}$ to the Square of the Sine of one half the Angle B.

But $\frac{AC+AM}{2}$ = 24 Degrees 2′, and $\frac{AC-AM}{2}$ = 18 Degrees 6′; and becauſe the firſt Term of the Analogy is the Rectangle under the Sines of A B, B C, and ſecond Term is the Square of Radius, the Sum of the Logarithm Sine of A B, B C, muſt be taken from double the Logarithm of Radius, and what remains muſt be added to the Sum of the Logarithm S, of $\frac{AC+AM}{2}$, and $\frac{AC-AM}{2}$, which is the ſame as if the Logarithm Sines of each of the Arcs A B, B C, were ſubtracted from the Log. of Radius; or if the Complements Arithmetical of theſe Sines be taken, and the Complements and the ſaid Sines be all added together, then ſhall the Sum be the Logarithm of the Square of the Sine of half the Angle B. And ſo the Half of the Logarithm 9.8965274 is the Logarithm Sine of half the Angle B = 51 Degrees 59′ 56″, and the Double of this Angle ſhall be 103 Degrees 59′ 52″ = B, which was ſought.

Log. S, B C Comp. Arith.	0.3010299
Log. S, A B Comp. Arith.	0.3898364
Log. S, $\frac{AC+AM}{2}$	9.6098803
Log. S, $\frac{AC-AM}{2}$	9.4923083
2 Log. S, Angle B.	19.7930549

CHAP. V.

Of the continual Increments of proportional Quantities, and how to find by Logarithms, any Term in a Series of Proportionals, either increaſing or decreaſing. Fig. 3.

IF any-where in the Axis of the Logarithmetical Curve, there be taken any Number of equal Parts SV, VY, YQ, *&c.* and at the Points S, V, Y, Q, *&c.* be raiſed the Perpendiculars ST, VX, YZ, QΠ, *&c.* then from the Nature of the Curve ſhall all theſe Perpendiculars be continually proportional; and therefore alſo the continual Increments X*x*, Z*z*, Ππ; ſhall be proportional to their Wholes: For ſince ST : VH : : VX : YZ : : YZ : QΠ, it ſhall be (by Diviſion of Proportion) ST : X*x* : : VX : Z*z* : : YZ Ππ; and (by Compoſition of Proportion) VX : X*x* : YZ : Z*z* : : QΠ : Ππ. Hence, if X*x* be any Part of any Right Line ST, then will Z*z* be the ſame Part of the Right Line VX, and alſo Ππ the ſame Part of the Right Line YZ. For Example; if X*x* be the $\frac{1}{20}$ Part of ST, then will Z*z* = $\frac{1}{20}$ VX, and Ππ = $\frac{1}{20}$ YZ; or, which comes to the ſame, we ſhall have VX = ST + $\frac{1}{20}$ ST, YZ = VX + $\frac{1}{20}$ VX. Alſo QΠ = YZ + $\frac{1}{20}$ YZ.

Now make, as ST is to VX, ſo is Unity AB to NR; then ſhall AN = SV; and ſo each of the Right Lines SV, VY, YQ, *&c.* ſhall be equal to the Logarithm of RN; and AV, the Logarithm of the Term VX, ſhall be equal to AS + AN = Logarithm of ST + Logarithm of NR. Alſo AY, the Logarithm of the Term YZ, ſhall be equal to AS + 2 AN = Logarithm ST + 2 Logarithm NR; and AQ, the Logarithm of the Term QΠ, ſhall be equal to AS + 3 AN = Logarithm ST + 3 Logarithm NR. And univerſally, if the Logarithm of the Number NR be multiplied by a Number expreſſing the Diſtance of any Term from the firſt, and the Product be added to the Logarithm of the firſt Term, then will the Logarithm of that Term be had: But if a Series of Proportionals be decreaſing, that is, if the Terms diminiſh in a continual Ratio, and QΠ be the firſt Term; then the Logarithm of any other will be had, in multiplying the Logarithm of the Number N R, by a Number that expreſſes the Diſtance of its Term from the firſt, and ſubtracting the Product from the Logarithm of the firſt. And if the ſaid Product be greater than the Logarithm of the firſt Term, then the Logarithms muſt begin from a Unit in ſome Place of Decimal Fractions, as from O P, and then the Logarithm of the Number QΠ will be O Q.

Now, let L M repreſent any Money, or Sum of Money, put out to Intereſt, ſo that the Intereſt thereof be accounted but at the End of every Year, and let K *k* be the Gain or Intereſt thereof at the End of the firſt Year; then will I K be the Sum of the Intereſt and Principal. And again, I K becoming the Principal at the End of the firſt Year, H *h*, which is proportional to I K, or in a conſtant Ratio, will be the Gain at the End of the ſecond Year; and ſo H G, at the End of the ſecond Year, will become the Principal; and at the End of the third Year F *f*, proportional to G H, will be the Gain. Now, let us ſuppoſe the Principal be augmented every Year $\frac{1}{20}$ Part thereof, ſo that $IK = LM + \frac{1}{20} LM$, $GA = IK + \frac{1}{20} IK$, $EF = GH + \frac{1}{20} GH$, and ſo on. And accordingly the Terms L M, I K, G H, E F, *&c.* continual Proportionals, it is required to find the Amount of the Money at the End of any Number of Years.

Let L M be a Farthing. Becauſe L M is to I K as 1 to $1 + \frac{1}{20}$, or as 1 to 1.05. as A B is to N R, then will $NR = 1.05$, whoſe Logarithm A N is 0.0211893, or more accurately 0.0211892991, it is required to find the Amount of a Farthing put out at compound Intereſt, at the End of 600 Years. Multiply A N by 600, and the Product will be 12.7135794, and to this Product add the Logarithm of the Fraction $\frac{1}{960}$. *viz.* 97.0177288, (for a Farthing is $\frac{1}{960}$ Part of a Pound) and the Sum 109.7313082 ſhall be the Logarithm of the Number ſought; and ſince the Index 109 exceeds the Index of Unity by 9, there ſhall be nine Places of Figures above Unity in the correſpondent Number; and that Number, being ſought in the Tables, will be found greater than 5386500000, and leſs than 5386600000. And therefore a Farthing put out at Intereſt upon Intereſt, at 5 *per Cent. per Annum*, at the End of 600 Years will amount to above 5386500000 Pounds; which Sum could hardly be made up by all the Gold and Silver that has been dug out of the Bowels of the Earth from the Beginning of the World to this Time.

Let QΠ expound any Sum of Money due to ſome Perſon at the End of a full Year. Now it is certain, that if the Debtor ſhould pay down preſent the whole Sum of Money, he would loſe the yearly Uſury or Intereſt that his Money would gain him; and ſo a leſſer Sum, being put out to Intereſt, will at the End of one Year, together with the Intereſt thereof, be equal to the Sum of Money QΠ. Now this preſent Sum of Money, which, together with the reſt thereof, is equal to the Sum of Money QΠ, is called the preſent Worth of the Money QΠ. Let AN be the Logarithm of the Ratio which the Principal has to the Sum of the Principal and Intereſt, that is, if the Principal be twenty times the yearly Intereſt, let AN be the Logarithm of the Number $1 + \frac{1}{20}$ or 1.05, and take QY equal to AN; then will AY be the Logarithm of the preſent Worth of the Money QΠ. For it is manifeſt, that the Money YZ put out to Intereſt, will at the End of one Year amount to the Money QΠ; and ſo to have the Logarithm of the preſent Worth thereof, or YZ, the Logarithm AN muſt be taken from the Logarithm AQ, and there will remain the Logarithm AY of the preſent Worth, or YZ. But if the Sum QΠ be not due till the End of two Years, then the Logarithm 2 AN muſt be ſubtracted from the Logarithm AQ, and there will remain AV, the Logarithm of the preſent Worth, or of the Sum that muſt be paid down preſent for the Money QΠ due at the End of two Years. For it is manifeſt, that the Money VX being put out to Intereſt, will, at the End of two Years, amount to the Sum of Money QΠ. By the ſame Reaſon, if the Sum QΠ be not due until the End of three Years, the Logarithm 3 AN muſt be ſubtracted from the Logarithm of QΠ and the Remainder AS ſhall be the Logarithm of the Number ST, or

ST ſhall be the preſent Worth of the Sum QΠ due at the three Years End. And univerſally, if the Logarithm AN be multiplied by the Number of Years, at the End of which the Sum QΠ is due, and the Number produced be taken from the Logarithm AQ, then will the Logarithm of the preſent Worth of the Sum QΠ be had. And from hence it is manifeſt, if 5386500000 Pounds be due to ſome Society at the End of 600 Years, then would the preſent Worth of that vaſt Sum of Money be ſcarcely a Farthing.

If the proportional Right Line HG, EF, AB, CD, *Fig.* 4. are Ordinates to the Axis of the Logarithmical Curve, and if their Ends FH, DB, be joined by Right Lines, which, produced, meet the Axis in the Points P and K, then the Right Lines GP, AK, will be always equal. For ſince GH : EF : : AB : CD, it will be as GH : F*s* : : AB : DR. But becauſe of the equiangular Triangles PGH, H*s*F, as alſo KAB, BRD, we have PG : H*s* : : (GH : F*s* : : AB : DR : :) KA : BR. And ſince the Conſequents H*s*, BR, are equal, the Antecedents PG, KA, ſhall be alſo equal. W. W. D.

If the Right Lines CD, EF, equally accede to AB, GH, ſo that the Point D at laſt may coincide with B, and the Point F with H, then the Right Lines DBK, FHP, which did cut the Curve before, will be changed into the Tangents BT, HV. And the Right Lines AT, GV, will be always equal to each other; that is, the Portion of the Axis AT, or GV, intercepted between the Ordinate and the Tangent, which is called the Subtangent, will every-where be a conſtant and given Length. And this is one of the chief Properties of the Logarithmical Curve; for the different Species or Forms of thoſe Curves are determined by the Subtangents.

The Logarithms, or the Diſtances from Unity of the ſame Number, in two Logarithmical Curves of different Species, will be proportional to the Subtangents of their Curves. For let HBD, SNY, *Fig.* 4, 5. be Curves, whoſe Subtangents are AT, MX, and let AB = MN = Unity; alſo DC = QY; then ſhall AC the Logarithm of the Number CD, in the Logarithmical Curve HD, be to MQ, the Logarithm of the Number QY, (or of the ſaid CD) in the Curve SY, as the Subtangent AT is to the Subtangent

MX. For let there be ſuppoſed an infinite Number of mean proportional Terms between AB, CD, or NM, QY, in the Ratio of AB to *ab*, or MN to *mn*; and ſince AB = MN, then will *ab* = *mn*, as alſo *bc* = *no*. And becauſe the Number of proportional Terms in each Figure are equal, they do divide the Lines AC, MQ, into equal Numbers of Parts, the firſt of which A*a*, M*m*, and ſo the ſaid Parts ſhall be proportional to their Wholes; that is, it will be as A*a* : M*m* : : AC : MQ. And becauſe the Triangles TAB, B*cb*, are ſimilar, (for the Part of the Curve B*b* nearly coincides with the Portion of the Tangent) as alſo the Triangles XMN, N*on*, we have A*a*, or B*c* : *bc* : : TA : AB.

Alſo as *no*, or *bc* : N*o* : : MN, or AB : MX.

Where (by Equality of Proportion) it will be B*c* : N*o* : : TA : MX : : A*a* : M*m* : : AC : MQ; *which was to be demonſtrated.* If AT be called *a*, ſince AB : AT : : *bc* : B*c*, then will $Bc = \frac{a \times bc}{AB}$.

Hence, if the Logarithm of a Number extremely near Unity, or but a ſmall matter exceeding it, be given, then will the Subtangent of the Logarithmical Curve be had. For the Exceſs *bc* is to the Logarithm B*c*, as Unity AB is to the Subtangent AT. Or even if there are any two Numbers nearly equal, their Difference ſhall be to the Difference of their Logarithms, as one of the Numbers is to the Subtangent. For Example, if the Increment b*c* be ,00000 00000 00001 02255 31945 60259, and B*c* or A*a* the Logarithm of the Number *ab* be ,00000 00000 00000 44408 92098 50062. Now if a fourth Proportional be found to the ſaid two Numbers and Unity, *viz.* 43429 4481903251, this Number will give the Length of the Subtangent AT, which is the Subtangent of the Curve expreſſing *Briggs*'s Logarithms.

If a Sum of Money be put out to Intereſt on this Condition, that a proportional Part of the yearly Rate of Intereſt thereof be accounted every Moment of Time, *viz.* ſo, that at the End of the firſt Moment of Time, or indefinitely ſmall Particle of a Year, the Intereſt gotten thereby be proportional to that Time; which being added to the Principal, again begets Intereſt at the End of the ſecond Moment of Time, and then the Principal and this Intereſt become a Principal, and ſo on; it is requir'd to find the Amount of that Sum at the Year's End. Let a be nearly the Intereſt of Unity, or of one Pound. Then, if one whole Year, or 1, gives the Intereſt a, the indefinitely ſmall Particle of a Year Mm will give the Intereſt $Mm \times a$, proportional to Mm; and accordingly, if Unity be expounded by MN, the firſt Increment thereof ſhall be $no = Mm \times a$. This being granted, let a Logarithmical Curve be ſuppos'd to be deſcribed through the Points Nn, whoſe Axis is OMQ. Then in this Curve, if the Portion of the Axis MQ expreſſes the Time, the Ordinate Qy will repreſent the Money proportionally increaſing every Moment, to that Time. For if there be taken ml, &c. $= Mm$, the Ordinates lp, &c. ſhall be in a Series of continual Proportionals in the Ratio of MN to mn; that is, they increaſe in the ſame Ratio as the Money doth.

Again, let the Right Line NX touch the Logarithmical Curve in N, and the Subtangent thereof MX ſhall be conſtant and invariable, and the ſmall Triangle Non ſhall be ſimilar to the Triangle XMN. But it has been prov'd, that the Increment $no = Mm \times a = No \times a$; and ſo $no : No :: No \times a : No :: a : 1$. But as no is to No, ſo ſhall NM be to MX. Wherefore it ſhall be as a is to 1, ſo is NM, or 1, to $MX = \frac{1}{a} =$ Subtangent.

Now if the nearly Rate of Intereſt be $\frac{1}{20}$ Part of the Principal, or if $a = \frac{1}{20} = .05$, then will $MX = \frac{1}{a} = 20$.

Becauſe of different Forms of Logarithms, the Logarithms of the ſame Number are proportional to the Subtangents of their Curves: If MQ expreſſes the Time of a whole Year, or Unity, then ſhall QY be the Amount of the Money at the Year's End. And to find QY, ſay, As MX, or $\frac{1}{20}$, is to 0.4342944, (which Number expounds the Subtangent of the Logarithmical Curve expreſſing *Briggs*'s Logarithms) ſo is one Year or Unity to a *Briggian* Logarithm, anſwering to the Number QY. This Logarithm will be found 0.0217147, and the Number anſwering to the ſame is $1.05127 = QY$, whoſe Increment above

Unity, or the Principal, exceeds the yearly Intereſt ,05 but a ſmall Matter. And ſo if the yearly Intereſt of 100 Pounds be 5 Pounds, the proportional yearly Intereſt, which is added to the Principal 100 at the End of each Particle of the Year, will amount only at the Year's End to 5 Pounds 2 Shillings and 6 $\frac{1}{2}$ Pence.

And if ſuch a Rate of Intereſt be requir'd, that every Moment a Part of it continually proportional to the increaſing Principal be added to the Principal, ſo that at the Year's End an Increment be produc'd that ſhall be any given Part of the Principal; for Example, the $\frac{1}{20}$ Part; ſay, As the Logarithm of the Number 1. 05 is to 1; that is, as 0.0211893 is to 1; ſo is the Subtangent 0. 432944 to $\frac{1}{a}=20.\ 49$, and then will $a=\frac{1}{20\ 49}=.0488$. For if ſuch a Part of the Rate of Intereſt .0488 be ſuppoſed, as anſwers to a Moment, that is, having the ſame Ratio to .0488 as a Moment has to a Year, and it be made as Unity is to that Part of the Rate of Intereſt, ſo is the Principal to the momentaneous Increment thereof; then will the Money, continually increaſing in that manner, be augmented at the Year's End the $\frac{1}{20}$ Part thereof.

CHAP. VI.

Of the Method by which Mr. Briggs *computed his Logarithms, and the Demonſtration thereof.*

ALthough Mr. *Briggs* has no-where deſcrib'd the Logarithmical Curve, yet it is very certain, that from the Uſe and Contemplation thereof, the Manner and Reaſon of his Calculations will appear. In any Logarithmical Curve H B D, let there be three Ordinates A B, *a b*, *q s*, nearly equal to one another; that is, let their Differences have a very ſmall Ratio to the ſaid Ordinates; and then the Differences of their Logarithms will be proportional to the Differences of the Ordinates. For ſince the Ordinates are nearly equal to one another, they will be very nigh to each other, and ſo the Part of the Curve Bs, intercepted by them, will almoſt coincide with a ſtrait Line; for it is certain, that the Ordinates may be ſo near to each other, that the Difference between the Part of the Curve, and the Right Line ſubtending it, may have to that Subtenſe, a Ratio leſs than any given Ratio. Therefore the Triangles Bcb, Brs, may be taken for Right-lin'd, and will be equiangular. Wherefore, as $sr : bc :: Br : Bc :: Aq : Aa$; that is, the Exceſſes of the Ordinates or Lines above the leaſt, ſhall be proportional to the Differences of their Logarithms. And from hence appears the Reaſon of the Correction of Numbers and Logarithms by Differences and proportional Parts. But if A B be Unity, the Logarithms of Numbers ſhall be proportional to the Differences of the Numbers.

If a mean Proportional be found between 1 and 10, or, which is the ſame thing, if the Square Root of 10 be extracted, this Root or Number will be in the middle Place between Unity and the Number 10, and the Logarithm thereof ſhall be $\frac{1}{2}$ of the Logarithm of 10, and ſo will be given. If again, between the Number before found, and Unity, there be found a mean Proportional, which may be done in extracting the Square Root of the ſaid Number, this Number, or Root, will be twice nearer to Unity than the former, and its Logarithm will be one Half of the Logarithm of that, or one Fourth of the Logarithm of 10. And if in this manner the Square Root be continually extracted, and the Logarithms biſected, you will at laſt get a Number, whoſe Diſtance from Unity ſhall be leſs than the $\frac{1}{1000000000000000}$ Part of the Logarithm of 10. And after Mr. *Briggs* had made 54 Extractions of the Square Root, he found the Number 1. 00000 00000 00000 12781 91493 20032 3442, and its Logarithm was 0. 00000 00000 00000, 05551 11512 31257 82702. Suppoſe this Logarithm to be equal to Aq or Br, and let qs be the Number found by extracting the Square Root; then will the Exceſs of this Number above Unity, *viz.* $rs =$, 00000 00000 00000, 12781 91493 20032 34.

Now, by means of theſe Numbers, the Logarithms of all other Numbers may be found in the following manner: Between the given Number (whoſe Logarithm is to be found) and Unity, find ſo many mean Proportionals, (as above) till at laſt a Number be gotten ſo little exceeding Unity, that there be 15 Cyphers next after it, and a like Number of ſignificative Figures after thoſe. Let this Number be *a b*, and let the ſignificative Figures with the Cyphers prefixed before them, denote the Difference *b c*. Then ſay, As the Difference *r s* is to the Difference *b c*, ſo is B *r* a given Logarithm, to B *c*, or A *a*, the Logarithm of the Number *a b*; which therefore is given. And if this Logarithm be continually doubled, the ſame Number of Times as there were Extractions of the Square Root, you will at laſt have the Logarithm of the Number ſought. Alſo by this way may the Subtangent of the Logarithmical Curve be found, *viz.* in ſaying, As *r s* : B *r* : : AB, or Unity : AT, the Subtangent, which therefore will be found to be 0.4342944819 03251; by which may be found the Logarithms of other Numbers; to wit, if any Number N M be given afterwards, as alſo its Logarithm, and the Logarithm of another Number ſufficiently near to N M be ſought, ſay, As N M is to the Subtangent X M, ſo is *n o* the Diſtance of the Numbers to N *o* the Diſtance of the Logarithms. Now, if N M be Unity = AB, the Logarithms will be had by multiplying the ſmall Differences *b c* by the conſtant Subtangent A T.

By this way may be found the Logarithms of 2, 3, and 7, and by theſe the Logarithms of 4, 8, 16, 52, 64, *&c.* 9, 27, 81, 243, *&c.* as alſo 7, 49, 343, *&c.* And if from the Logarithm of 10 be taken the Logarithm of 2, there will remain the Logarithm of 5; ſo there will be given the Logarithms of 25, 125, 625, *&c.*

The Logarithms of Numbers compounded of the aforeſaid Numbers, *viz.* 6, 12, 14, 15, 18, 20, 21, 24, 28, *&c.* are eaſily had by adding together the Logarithms of the component Numbers.

But ſince it was very tedious and laborious to find the Logarithms of the prime Numbers, and not eaſy to compute Logarithms by Interpolation, by firſt, ſecond and third, *&c.* Differences, therefore the great Men, Sir *Iſaac Newton*, *Mercator*, *Gregory*, *Wallis*, and laſtly, Dr. *Halley*, have publiſhed infinite converging Series, by which the Logarithms of

Numbers to any Number of Places may be had more expediently and truer: Concerning which Series Dr. *Halley* has written a learned Tract, in the *Philoſophical Tranſactions*, wherein he has demonſtrated thoſe Series after a new Way, and ſhews how to compute the Logarithms by them. But I think it may be more proper here to add a new Series, by means of which may be found eaſily and expeditiouſly the Logarithms of large Numbers.

Let z be an odd Number, whoſe Logarithm is ſought; then ſhall the Numbers $z-1$ and $z+1$ be even, and accordingly their Logarithms, and the Difference of the Logarithms, will be had, which let be called y: Therefore alſo the Logarithm of a Number, which is a Geometrical Mean between $z-1$ and $z+1$, will be given, *viz.* equal to the half Sum of the Logarithms. Now the Series

$$y \times \frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z} + \frac{181}{15120z} + \frac{13}{25200z^9}, \text{ \&c.}$$

ſhall be equal to the Logarithm of the Ratio, which the Geometrical Mean between the Numbers $z-1$ and $z+1$, has to the Arithmetical Mean, *viz.* to the Number z.

If the Number exceeds 1000, the firſt Term of the Series $\frac{y}{4z}$ is ſufficient for producing the Logarithm to 13 or 14 Places of Figures, and the ſecond Term will give the Logarithm to 20 Places of Figures. But if z be greater than 10000, the firſt Term will exhibit the Logarithm to 18 Places of Figures; and ſo this Series is of great Uſe in filling up the Logarithms of the Chiliads omitted by *Briggs*. For Example; It is required to find the Logarithm of 20001. The Logarithm of 20000 is the ſame as the Logarithm of 2 with the Index 4 prefix'd to it; and the Difference of the Logarithms of 20000 and 20002, is the ſame as the Difference of the Logarithms of the Numbers 10000 and 10001, *viz.* 0.00004 34272 7687. And if this Difference be divided by $4z$, or 80004, the Quotient $\frac{y}{4z}$

ſhall be - - - - - - - 0. 00000 00005 42813
And if the Logarithm of the Geometrical Mean be added to the Quotient, the Sum will — 4. 30105 17093 02416
— 4. 30105 17098 45231

be the Logarithm of 20001. Wherefore it is manifeſt, that to have the Logarithm to 14 Places of Figures, there is no Neceſſity of continuing out the Quotient beyond ſix Places of Figures. But if you have a Mind to have the Logarithm to 10 Places of Figures only, as they are in *Vlaq*'s Table, the two firſt Figures of the Quotient are enough. And if the Logarithms of the Numbers above 20000 are to be found by this Way, the Labour of doing them will moſtly conſiſt in ſetting down the Numbers.

Note, This Series is eaſily deduced from that found out by Dr. *Halley*; and thoſe who have a Mind to be inform'd more in this Matter, let them conſult his above-nam'd Treatiſe.

next the Appendix

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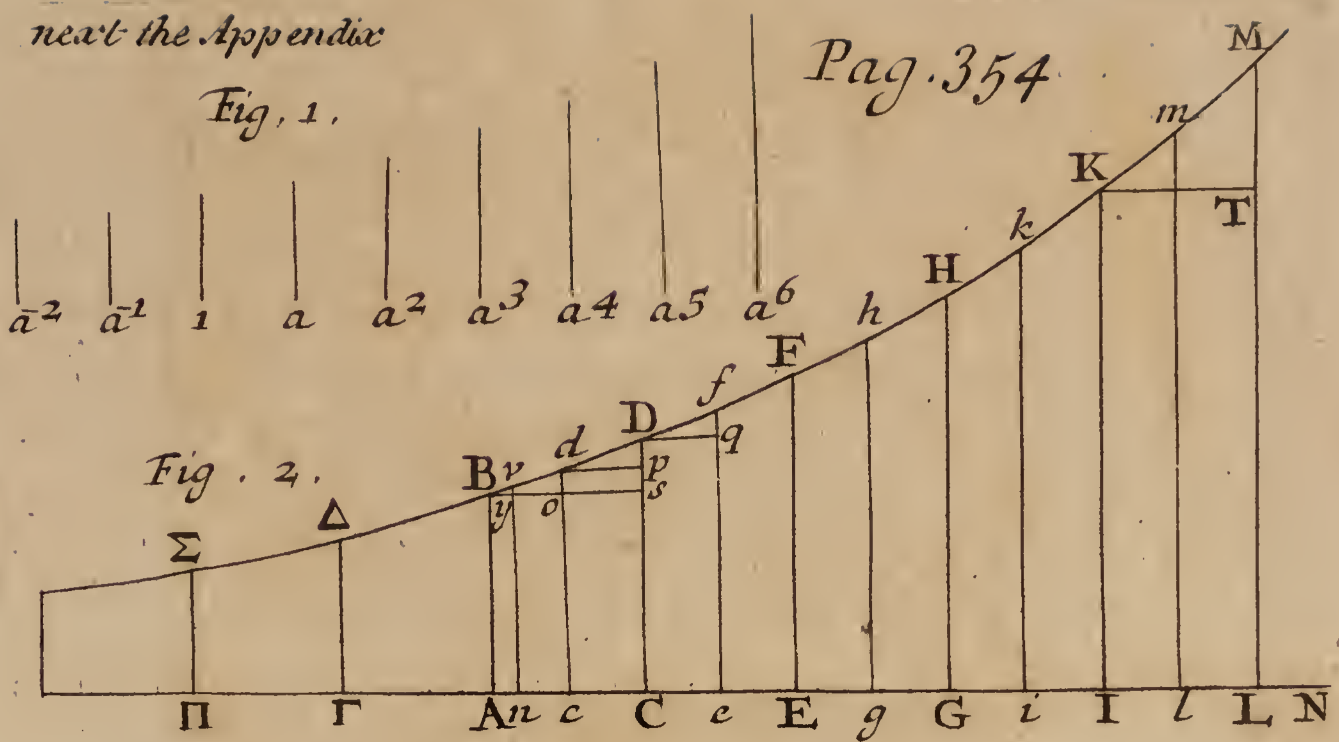
Fig. 1.

Fig. 2.



Chap. 4.

Fig. 3.

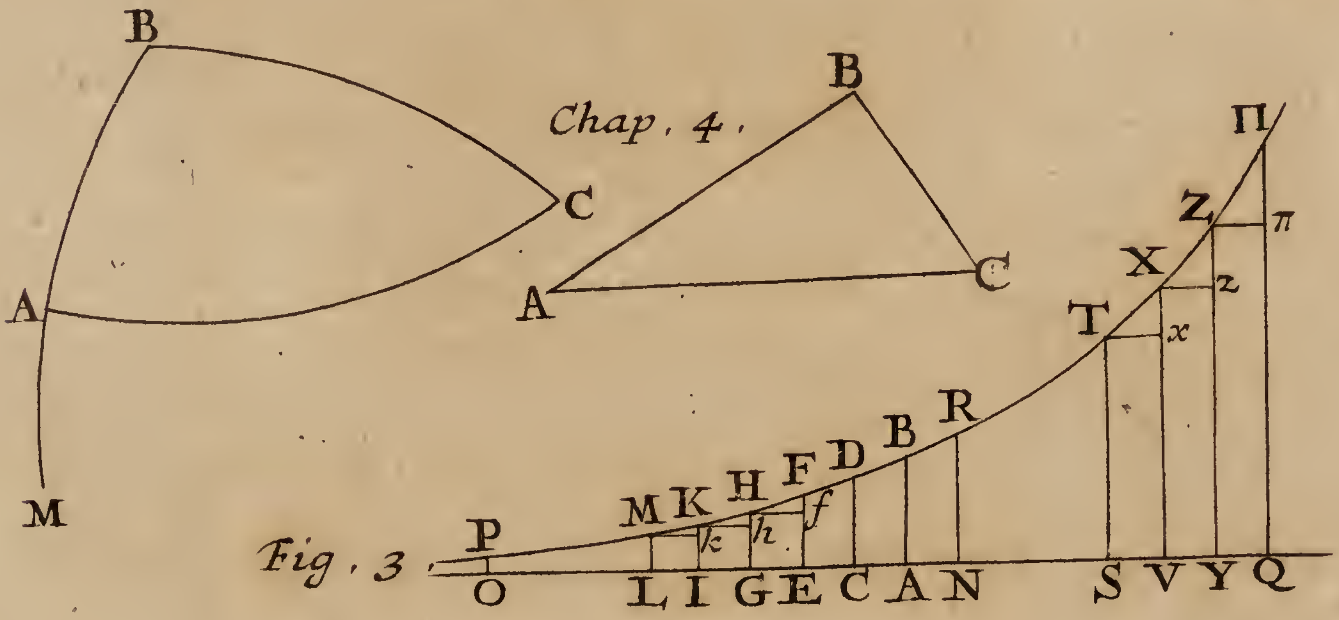


Fig. 4.

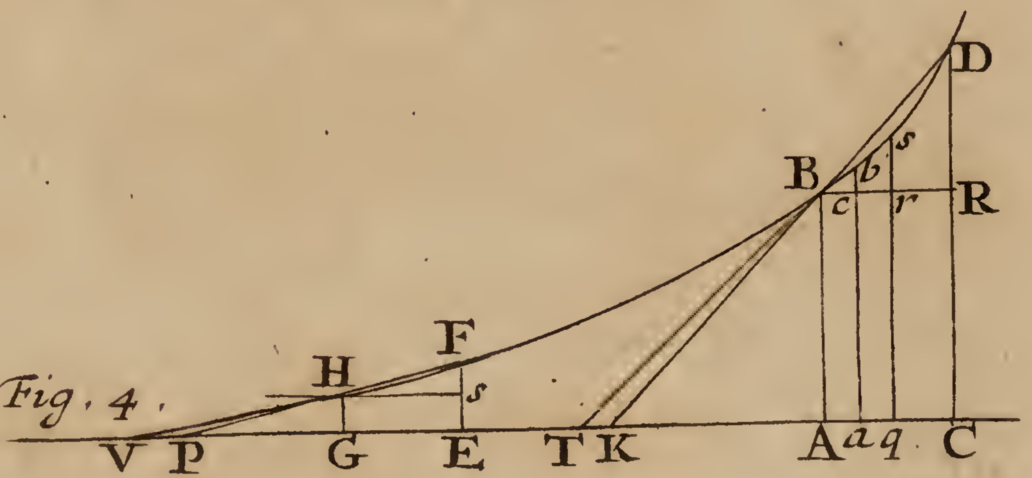
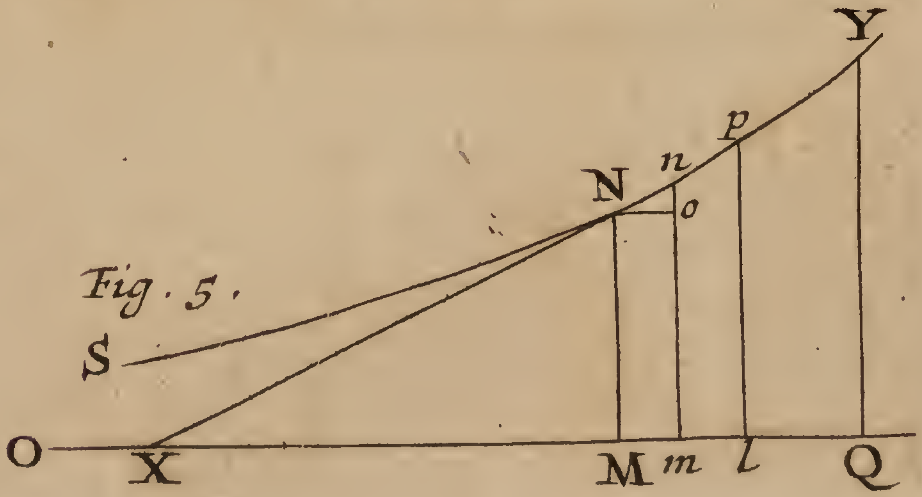
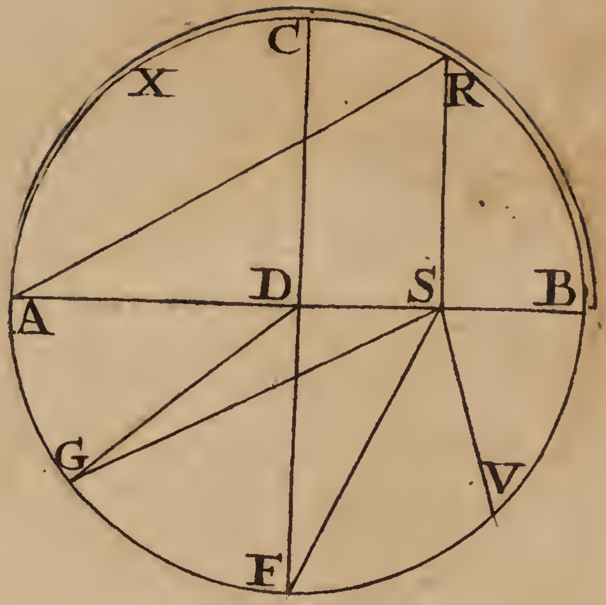


Fig. 5.

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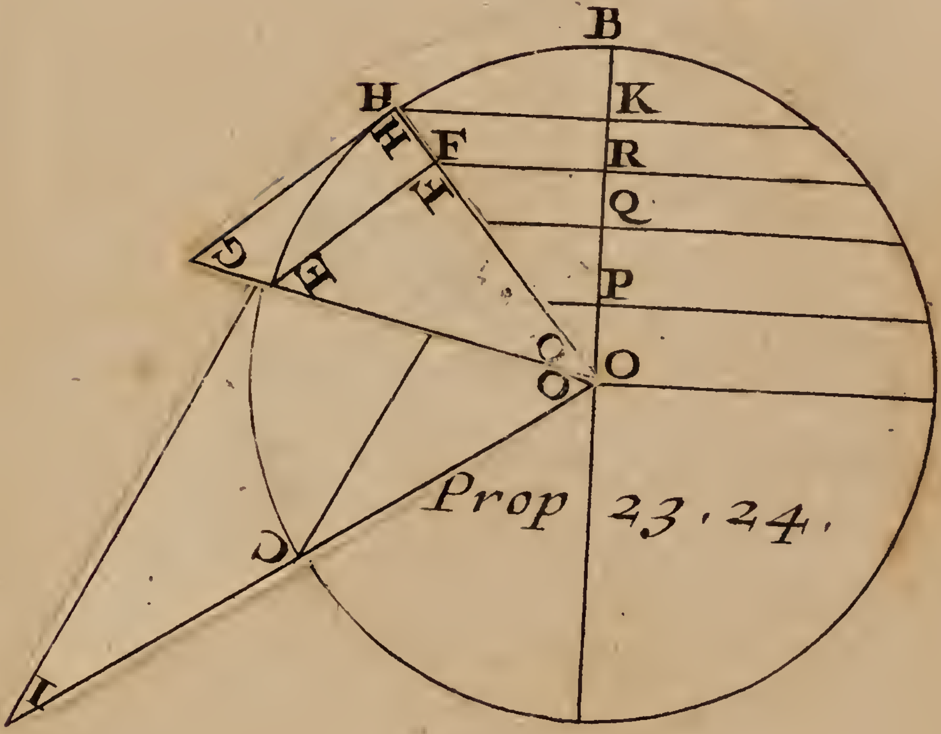
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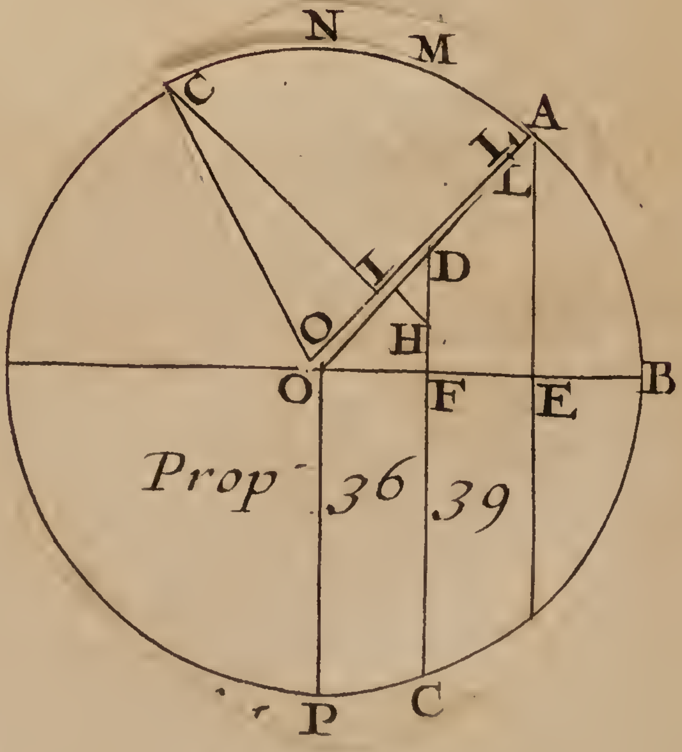
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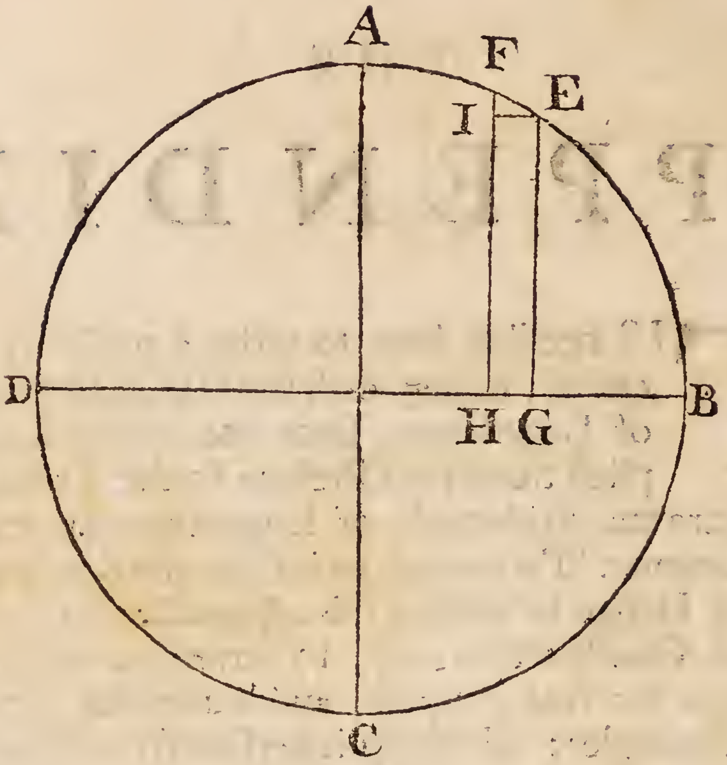
APPENDIX.

'TIS needleſs here to write a prefatory Diſcourſe, ſetting forth the Uſe and Invention of Logarithms, ſince the Author has ſupplied that in his Preface to the Treatiſe of the Nature and Arithmetic of Logarithms annex'd to theſe Elements: 'Tis enough to inform the Reader, that my chief Deſign in writing this *Appendix* was, to render their Conſtruction eaſy, by inveſtigating various Theorems for that Purpoſe, and illuſtrating them by proper Examples; all which is perform'd in the actual Operation of making the Logarithms of the firſt 10 Numbers, and of the prime Number 101, which is more than ſufficient to inform the meaneſt Capacity how to examine or conſtruct the whole Table. I have alſo ſhewn how, from the Logarithms given, to find its correſponding Number; and the Inveſtigation of the Series omitted by the Author in Page 355, for expeditiouſly finding the Logarithms of large Number. As to thoſe Series exhibited by him in his Trigonometrical Treatiſe, Page 287, for making the Sines and Coſines; I muſt declare, that I have exceeded my firſt Intentions, which were to give their Inveſtigation only; but conſidering, that as they depended upon the *Newtonian* Series, without the Inveſtigation of which, our Author's Series could never be thoroughly underſtood; I thought it would therefore prove acceptable, if I ſhew'd their Inveſtigations too, from which thoſe of our Author's eaſily flow. In order to which, and to keep the Reader no longer in Suſpenſe; let r be put for the Radius of the Circle ABCD; a, for the Arch BE, whoſe Length is to be inveſtigated; S, equal to the Sine; and v, the verſed Sine of that Arch; then is $FE = a$, $IF = s$, and $IE = GH = v$.



Whence $\dot a\dot a = \dot s\dot s + \dot v\dot v$; but from the Equation of the Curve, *viz.* $2rv - vv = ss$ we have $v = r - \sqrt{rr-ss}$.

Therefore $\dot v = \frac{\dot s\dot s}{rr-ss}\,\frac{1}{2}$ and $\dot v\dot v = \frac{{}^2\dot s\dot s}{rr-ss}$; which being ſubſtituted for $\dot v\dot v$ in the firſt Equation, we have $\dot a = \overline{\frac{s^2\dot s\dot s}{rr-ss} + \dot s\dot s = \frac{{}^2rr\dot s\dot s}{rr-ss}}^{\frac{1}{2}} = \frac{r\dot s}{rr-ss^{\frac{1}{2}}} = r\dot s \times \overline{rr-ss}^{-\frac{1}{2}}$.

But $\overline{rr-ss}^{-\frac{1}{2}}$ by Sir *Iſaac Newton*'s Binomical Theorem is $r^{-1} + \frac{1}{2}s^2r^{-3} + \frac{3}{8}s^4r^{-5} + \frac{15}{48}s^6r^{-7}$, *&c.* Wherefore $r\dot s \times \overline{rr-ss}^{\frac{1}{2}} = \dot s + \frac{s^2\dot s}{2r^2} + \frac{3s^4\dot s}{8r^4} + \frac{15s^6\dot s}{48r^6}$, *&c.* and the Fluent thereof, *viz.* $s + \frac{s^3}{6r^2} + \frac{3s^5}{40r^4} + \frac{15s^7}{336r^6}$, *&c.* $= s + \frac{s^3}{2.3r^2} + \frac{3s^5}{2.4.5r^4} + \frac{3.5s^7}{2.4.6.7r^6}$, *&c.* or $s + \frac{s}{2.3r^2} + \frac{3.3s^5}{2.3.4.5r^4} + \frac{3.3.5.5s^7}{2.3.4.5.6.7r^6}$, *&c.* is equal to the Arch of a Circle whoſe Radius is r, and Sine s. But if r be put equal to Unity, then $s + \frac{1}{2.3}s^3 + \frac{3.3}{2.3.4.5}s^5 + \frac{3.3.5.5}{2.3.4.5.6.7}s^7$, &c. will expreſs the Length of the Arch a.

EXAMPLE.

Let it be required to find the Length of the Arch of 30 Degrees, to 6 Places of Decimals, the Radius being Unity.

Here $s = \frac{1}{2}$, and $ss = \frac{1}{4}$ whence the Operation may be as follows:

$$\begin{array}{lr} S = & ,5000000 \\ S^3 = & ,1250000 \\ S^5 = & 312500 \\ S^7 = & 78125 \\ S^9 = & 19531 \\ S^{11} = & 4882 \\ S^{13} = & 1220 \\ S^{15} = & 305 \end{array} \qquad \left| \quad \begin{array}{lr} S = & ,5000000 \\ \frac{1}{6}S^3 = & 208333 \\ \frac{3}{40}S^5 = & 23437 \\ \frac{5}{112}S^7 = & 3487 \\ \frac{35}{1152}S^9 = & 593 \\ \frac{63}{2}{}_{1}\,S^{11} = & 109 \\ \frac{231}{13312}S^{13} = & 21 \\ \frac{143}{10240}S^{15} = & 4 \\ \hline & ,523598.4 \end{array} \right.$$

Hence the Length of the Arch of 30 Degrees is ,523598+. Now if this Arch be multiplied by 6, we ſhall have the Length of the Arch of the Semicircle in ſuch Parts as the Radius is 1, or of the whole Circumference in ſuch Parts as the Diameter is 1, *viz.* 3,14159+.

But there is no Series ſo eaſy to be retain'd in the Memory, and ſo readily put in Practice, for obtaining the Ratio of the Diameter of the Circle to its Circumference, as that which is deriv'd from the Tangent firſt given. For if t be put equal to the Tangent of any Arch, then $a = t\frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7 + \frac{1}{9}t^9$, &c.

Now the Radius being Unity, the Sine of 30 Degrees $= \frac{1}{2}$, and conſequently the Coſine $= \sqrt{\frac{3}{4}}$; and becauſe the Coſine is to the right Sine, as the Radius to the Tangent, it will be $\sqrt{\frac{3}{4}} : \sqrt{\frac{1}{4}} :: 1 : \sqrt{\frac{1}{3}}$, the Tangent of 30° $00^{1} = t$, whence $tt = \frac{1}{3}$; wherefore, if the Root of $\frac{1}{3}$ be divided continually by 3, and the ſeveral Quotients by all the odd Powers ſucceſſively, *viz.* the firſt by three, the ſecond by 5, *&c.* the Sum of the affirmative Quotients made leſs by all the negative ones, will be the Arch of 30 Degrees.

And becauſe the Arch of 30 Degrees is $\frac{1}{6}$ part of the Semi-circumference, if inſtead of $\sqrt{\frac{1}{3}}$ be taken 6 $\sqrt{\frac{1}{3}} = \sqrt{12}$. we ſhall have the Semi-circumference in ſuch Parts as the Radius is Unity; or the whole Circumference, the Diameter being Unity.

The OPERATION ſtands thus:

$\sqrt{12}$=3,494101		+ 3,464101	
3)	1154700		(— ,384900
5)	384900	(+ 76980	
7)	128300		(— 18328
9)	42766	(+ 4751	
11)	14255		(— 1295
13)	4751	(+ 365	
15)	1583		(— 105
17)	527	(+ 31	
19)	175		(— 9
		+ 3,546228	— ,404637

Whence 3,546228 — ,404637 = 3,14159,1 the ſame as before. The Impoſſibility of expreſſing the exact Proportion of the Diameter of a Circle to its Circumference by any receiv'd way of Notation, has put the moſt celebrated Men in all Ages upon approximating the Truth as near as poſſible; there being a Neceſſity of a near Quadrature, inaſmuch as it is the Baſis upon which the moſt uſeful Branches of the Mathematics are built. And after the famous *Van Ceulen*, who carried it to 36 Places of Decimals, which he order'd to be engraven on his Tomb-ſtone, thinking he had ſet Bounds to farther Improvements, the firſt that attempted it with Succeſs was the moſt indefatigable Mr. *Abraham Sharp*, who by a double Computation, *viz.* from the Sine of 6 Degrees one way, and from the Sine and Coſine of 12 Degrees another way, carried it to twice the Number of Places that *Van Ceulen* had done, *viz.* 72.

And ſince that Time Mr. *Machin*, the preſent Profeſſor of Aſtronomy in *Greſham-College*, and Secretary to the Royal Society, by a different Method of Computation, has carried it to 100 Places, almoſt triple the

Number that *Van Ceulen* had done; which not only confirms Mr. *Sharp*'s Quadrature, but ſhews us, that if the Diameter be 100000, *&c.* the Circumference will be 3,14159. 26535. 89793. 23846. 26433. 83279. 50288. 41971. 69399. 37510. 58209. 74944. 59230. 78164. 05286. 20899. 86280. 34825. 34211. 70679 × of the ſame Parts.

Which is a Degree of Exactneſs far ſurpaſſing all Imagination, being by Eſtimation more than ſufficient to calculate the Number of Grains of Sand that may be comprehended within the Sphere of the fixed Stars.

The late Mr. *Cunn*'s Series for determining the Periphery of an Ellipſis (who was my Predeceſſor in the Mathematical School erected by *Frederic Slare*, M. D. and eſtabliſh'd by a Decree of the High Court of Chancery for qualifying Boys for the Sea-Service) being new and curious, this Opportunity is taken of making it public.

Let A be equal to a Quadrant of the Circle circumſcribing the Ellipſis, whoſe Periphery is requir'd. Then $A \times$

$$1 - \frac{1.1}{2.2}e^2 - \frac{1.3.1}{2.4.8}e^4 - \frac{1.3.5.1}{2.4.6.16}e^6 - \frac{1.3.5.7.5}{2.4.6.8.128}e^8 - \frac{1.3.7.9.7}{2.4.6.8.10.256}e^{10} - \frac{1.3.5.7.9.11.21}{2.4.6.8.10.12.1024}e^{12}, \&c.$$

is the Periphery of a Quadrant of the Ellipſis where $ee = \frac{tt - cc}{tt}$ t being the Semi-tranſverſe Diameter, and c the Semi-conjugate.

When this Series came to hand, it was imperfect, inaſmuch as there were only the firſt five Terms without the Law of Continuation: But being deſirous of rendring it complete, after ſome Conſideration I found the Law to be as follows: It is plain by Inſpection, that the Numerators and Denominators of each Term are compos'd of Numbers that run in Arithmetical Progreſſion, except the laſt in each Term, *viz.* $\frac{1}{2}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{5}{128}$, *&c.* and thoſe being found by the continual Multiplication of theſe Fractions, $\frac{1}{1} \times \frac{1}{2} \times \frac{1}{4} \times \frac{3}{6} \times \frac{5}{8} \times \frac{7}{10} \times \frac{9}{12}$, *&c.* the Law of continuing the whole Series as above, is evident. Whence, by a well known Method of ſubſtituting Capital Letters for each *Term reſpectively, the following Series is deduc'd,* viz. $A \times 1 - \frac{1}{4}ee - \frac{1.3}{4.4}e^2$

$B - \frac{3.5}{6.6}e^2C - \frac{5.7}{8.8}e^2D - \frac{7.9}{10.10}e^2E - \frac{9.11}{12.12}e^2F$, *&c.* where the Law of Continuation is evident alſo, ſince each Capital Letter is equal to its precedent Term, *viz.* $B = \frac{1}{4}ee$, $C = \frac{1.3}{4.4}e^2B$, *&c.* and without Doubt in Practice is preferable to the former Series: But the Inveſtigation of that, on which this laſt depends, is omitted; purely on account of its being foreign to the preſent Subject.

But to return; if the Series expreſſing the Length of the Arch, *viz.* $s + \frac{1}{6}s^3 + \frac{3}{40}s^5$, *&c.* be revers'd, we ſhall have the Value of s in the Terms of a, and conſequently a direct Method for finding the Sine of any Arch from its Length given. Thus,

If $a = s + \frac{1}{6}s^3 + \frac{3}{40}s^5 + \frac{5}{112}s^7$ *&c.*

Then $s = a - \frac{1}{6}a^3 + \frac{1}{120}a^5 - \frac{1}{5140}a^7$, *&c.*

Or $s = a - \frac{a^3}{2.3} + \frac{a^5}{2.3.4.5} - \frac{a^7}{2.3.4.5.6.7}$, *&c.*

$$\left.\begin{array}{ll} \text{For put } s = & Aa + Ba^3 + Ca^5, \text{ \&c.} \\ \text{Then } \frac{1}{6}s = & \frac{1}{6}A^3a^3 + \frac{1}{2}A^2, Ba^5, \text{ \&c.} \\ \text{And } \frac{3}{40}s^3 = & \frac{3}{40}A^5, a^5, \text{ \&c.} \end{array}\right\} = a.$$

And conſequently $Aa = a$, and $A = 1$, alſo $B + \frac{1}{6}A = 0$, and $B = -\frac{1}{6}$, $A^3 = -\frac{1}{6}$, alſo $C + \frac{1}{2}A^2B + \frac{3}{40}A^5 - 0$, and $C = -\frac{1}{2}A^2B - \frac{3}{40}A^5 = -\frac{1}{2} \times -\frac{1}{6} - \frac{3}{40} = \frac{1}{12} - \frac{3}{40} = \frac{1}{120}$. Wherefore $A = 1$, $B = -\frac{1}{6}$, $C = \frac{1}{120}$, *&c.* and conſequently, $s = a - \frac{a}{6} + \frac{a^5}{120}$, *&c.* From which three Terms the Law of Continuation is eaſily diſcover'd, making $s = a - \frac{a^3}{2.3} + \frac{a^5}{2.3.4.5} - \frac{a^7}{2.3.4.5.67.} + \frac{a^9}{2.3.4.5.6.7.8.9}$, *&c* Whence ſubſtituting A for a, and we ſhall have $A - \frac{A^3}{1.2.3} + \frac{A^3}{1.2.3.4.5} - \frac{A}{1.2.3.4.5.6.7} + \frac{A}{1.2.3.4.5.6.7.8.9}$, *&c.* for the *Newtonian* Series, according to our Author's Form, for finding the Sine of any Arch, its Length being given. Q. E. I.

Again, becauſe the Square of the Radius made leſs by the Square of the Sine, is equal to the Square of the Coſine, by the ſecond Propoſition of our Author's Elements of plain Trigonometry; it follows, that if from the Square of the Radius $=1$ be taken the Square of the Sine $=a-\frac{1}{6}a^3+\frac{1}{120}a^5$, *&c.* the Square Root of the Remainder, will be the Coſine $=1-\frac{1}{2}a^2+\frac{1}{24}a^4-\frac{7}{720}a^6$ *&c.* thus,

$$\begin{array}{rl}
S= & a-\frac{1}{6}a+\frac{1}{120}a^5,\ \&c. \\
S= & a-\frac{6}{1}a^3+\frac{1}{120}a^5,\ \&c. \\ \hline
 & a^2-\frac{1}{6}a^4+\frac{1}{120}a^6,\ \&c. \\
 & \quad -\frac{1}{6}a^4+\frac{1}{35}a^6,\ \&c. \\
 & \qquad +\frac{1}{120}a^6,\ \&c. \\ \hline
\end{array}$$

$Ss=a^2-\frac{1}{3}a^5+\frac{2}{45}a^6$, *&c.* which being taken from the Square of the Radius 1, leaves $1-aa+\frac{1}{3}a^5-\frac{2}{45}a^6$, *&c.* the Square Root of which will be the Coſine.

$$1-aa+\frac{1}{3}a^5-\frac{2}{45}a^6,\ \&c.\ \left(1-\frac{1}{2}a^2+\frac{1}{24}a^4-\frac{1}{720}a^5,\ \&c.\right.$$

$$\begin{array}{rl}
 & 1 \\ \hline
2-aa) & -aa+\frac{1}{3}a^4 \\
 & -aa+\frac{1}{4}a^4 \\ \hline
2-aa,\ \&c.) & \frac{1}{12}a^4-\frac{2}{45}a^6,\ \&c. \\
 & \frac{1}{12}a^4-\frac{1}{24}a^6,\ \&c. \\ \hline
2-aa,\ \&c.) & -\frac{1}{30}a^6,\ \&c. \\
 & \frac{1}{30}a^6,\ \&c. \\ \hline
 & 0
\end{array}$$

Wherefore putting A for *a*, we ſhall have for the Coſine $1-\frac{A^2}{2}+\frac{A^4}{24}-\frac{A^6}{720}$, *&c.* or $1-\frac{A^2}{1.2}+\frac{A^4}{1.2.3.4.}-\frac{A^6}{1.2.3.4.5.6.}+\frac{A^8}{1.2.3.4.5.6.7.8.}$ *&c.* according to the Author's Form. Q. E. I.

But becauſe thoſe Series, as our Author obſerves, converge very ſlowly, eſpecially when the Arch is nearly equal to the Radius, he therefore deviſed (*Page* 287) other Series, whoſe Inveſtigation may be as follows:

Let the Arc, whoſe Sine is ſought, be the Sum or Difference of two Arcs, *viz.* $A+z$, or $A-z$: And let the Sine of the Arc A be called a, and the Coſine b.

Now, if the Arc $DF = DE$, Prop. 5*th* of the Elements of Trigonometry be call'd z, then its Sine FO, will by the *Newtonian* Series be $= z - \frac{z^3}{1.2.3} + \frac{z^5}{1.2.3.4.5} - \frac{z^7}{1.2.3.4.5.6.7}$, *&c.* and its Coſine CO $= 1 - \frac{z^2}{1.3} + \frac{z^4}{1.2.3.4} - \frac{z^6}{1.2.3.4.5.6}$, *&c.* and becauſe $CD : DK :: CO : OP$. Therefore $OP = a - \frac{az^2}{1.2.} + \frac{az^4}{1.2.3.4} - \frac{az^6}{1.2.3.4.5.6}$, *&c.*

Again, becauſe the Triangles CDK, FOM, are ſimilar, it will be as $CD : CK :: FO : FM$; whence $FM = \frac{bz}{1} - \frac{bz^3}{1.2.3} + \frac{bz^5}{1.2.3.4.5} - \frac{bz^7}{1.2.3.4.5.6.7}$, *&c.*

But $OP + FM = IF$, the Sine of the Arc BF, *viz.* $A+z$; conſequently the Sum of thoſe Series, *viz.* $a + \frac{bz}{1} - \frac{az^2}{1.2.} - \frac{bz^3}{1.2.3} + \frac{az^4}{1.2.3.4} + \frac{bz^5}{1.2.3.4.5}$, *&c.* is the Sine of the Arc $A+z$. And becauſe $FM = MG$, therefore their Difference $a - \frac{bz}{1} - \frac{az^2}{1.2} + \frac{bz^3}{1.2.3} + \frac{az^4}{1.2.3.4} - \frac{bz^5}{1.2.3.4.5}$, *&c.* is the Sine of the Arc $A-z$, *viz.* EL.

And again, becauſe $CD : CK :: CO : CP$; therefore $CP = b - \frac{bz^2}{1.2} + \frac{bz^4}{1.2.3.4} - \frac{bz^6}{1.2.3.4.5.6} + \frac{bz^8}{1.2.3.4.5.6.7.8}$, *&c.* And by reaſon of the ſimilar Triangles CDK, FMO, it will be as $CD : DK :: FO : MO$. Whence $MO = az - \frac{az^3}{1.2.3} + \frac{az^5}{1.2.3.4.5} - \frac{az^7}{1.2.3.4.5.6.7}$, *&c.*

But $CP - MO = CI$, the Coſine of the Arc $A+z$.

Wherefore the Coſine of the Arc $A+z$ is $b-\frac{az}{1}-\frac{bz^2}{1.2}+\frac{az^3}{1.2.3}+\frac{bz^4}{1.2.3.4}-\frac{az^5}{1.2.3.4.5}-\frac{bz^6}{1.2.3.4.5.6}$ &c.

And becauſe $MO=PL$, therefore $CP+MO=CL$, and conſequently the Coſine of the Arc $A-z=b+\frac{az}{1}-\frac{bz^2}{1.2}-\frac{az^3}{1.2.3}+\frac{bz^4}{1.2.3.4}+\frac{az^5}{1.2.3.4.5}$, &c. Q. E. I.

Now the Arc A is an Arithmetical Mean between the Arcs $A-z$ and $A+z$, and the Difference of their Sines are $\frac{bz}{1}-\frac{az^2}{1.2}-\frac{bz^3}{1.2.3}+\frac{az^4}{1.2.3.4}+\frac{bz^5}{1.2.3.4.5}-\frac{az^6}{1.2.3.4.5.6}$, &c. $\frac{bz}{1}+\frac{az^2}{1.2}-\frac{bz^3}{1.2.3}-\frac{az^4}{1.2.3.4}+\frac{bz^5}{1.2.3.4.5.}+-\frac{az^6}{1.2.3.4.5.6}$, &c. Whence the Difference of the Differences, or ſecond Difference is $\frac{2az^2}{1.2}-\frac{2az^4}{1.2.3.4}+\frac{2az^6}{1.2.3.4.5.6}$, &c. or $2a\times\frac{z^2}{1.2}-\frac{z^4}{1.2.3.4}+\frac{z^6}{1.2.3.4.5.6}$ &c. Which Series is equal to double the Sine of the mean Arc, drawn into the verſed Sine of the Arc z, and converges very ſoon; ſo that if z be the Arch of the firſt Minute of the Quadrant; our Author ſays the firſt Term of the Series gives the ſecond Difference to 15 Places of Figures, and the ſecond Term to 25 Places.

Whence, the following Rule is deriv'd for finding the Sine of the Arc $A+z$, or $A-z$.

RULE.

From double the Sine of the mean or middle Arc, ſubtract the ſecond Difference found by the Theorem; and from the Remainder, ſubtract the Sine of the given Extreme, whether it be the greater or leaſt, and the Remainder will be the Sine of the other Extreme.

EXAMPLE.

Let it be required to find the Sine of 30° 01′, the Sines of 30° 00′, and 29° 59′ being both given.

Here 30° 00′ is the mean Arc, whoſe Sine is 500000 00000, and the Sine of 29° 59′, the given Extreme, is 4,9974806226, and the Length of the Arc z, *viz.* one Minute, is ,000 29 0888208; which ſquar'd and multiplied by the Sine of the mean Arc, 50000, *&c.* according to the Direction of the Theorem, the Product will be the ſecond Difference, equal to ,000000042307; which ſubtracted from double the Sine of the mean Arc, equal to 1, the Remainder will be ,'999999957693; from which ſubtract the Sine of the given Extreme (which in this Caſe is the leaſt) and there will remain ; 5002518943 for the Sine of 30° 01′, the greater Extreme.

This Method of making the Sines, however it may appear at firſt Sight, is ſo far from being tedious or troubleſome, that I look upon it to be the moſt eligible of any other whatſoever; for the Square of z being once determined, and the ſeveral Multiples of it by the nine Digits made, and ſet down in a Table orderly, all the Sines may be made by Addition and Subtraction only; as indeed our Author hints they may by the Method demonſtrated in the 10th Propoſition of the Elements of Trigonometry; but this is evidently preferable to that, tho' a good Method too; and by which, all the Sines of the Quadrant, I preſume, were wont to be made, at leaſt as far as 30, or 60 Degrees; for after the Sines as far as 60 Degrees are obtained, all the others may be had by Addition only; and notwithſtanding there are other excellent Theorems, which contribute very much towards finiſhing and confirming the Truth of the whole Canon; yet this deduced from our Author's Series, I deem the moſt elegant and fit for Practice; becauſe the Difference of the Differences of the Sines being what is always required to be found, there will be ſeven Cyphers at leaſt before the ſignificant Figures of the ſaid Difference; which is the Product made by the Square of z, into the Sine of the mean Arc: So that to have the Sine true to ten Places, there will not be occaſion to find above four or five Figures in the Product, which, according to the common Method of contracted Multiplication, may be obtained with very few Figures. Thus, for Inſtance, the Sine of 30° 02′ may be had to ten Places by a wonderful eaſy Operation, the Sines of 30° 01′ and 30° 00′ being both given.

EX-

EXAMPLE.

The Sine of 30° 01′ is	,50025189543
The Square of z inverted	16480000000
	40020
	2001
	300
	5
	42326

Whence the Product is ,000000042326 true to eleven Places at leaſt. Wherefore if, according to the Rule, from

double the Sine of the middle Arc =	1,00050379086
we ſubtract the ſaid Product,	,00000004232
And from the Remainder	1,00050374854
the Sine of 30° 00′ the given Extreme	,50000000000
be ſubtracted	,50050374854

There will remain ,50050374854 for the Sine of 30° 02′ the other Extreme; than which, nothing of this Nature can be deſired more eaſy.

SCHOLIUM.

Becauſe the Difference of the Differences of the Sines, or ſecond Difference, has always 7 Cyphers before the ſignificant Figures; it follows, that the whole Canon, where the Sines conſiſt but of 6 Places, which is as far as our Tables for common Practice need extend, may be perform'd chiefly by Addition and Subtraction only, without forming Multiples of the Square of z by the nine Digits; tho' perhaps it may be neceſſary to uſe the Method of contracted Multiplication every 5th Minute to confirm the Truth, leſt in continual doubling and ſubtracting, an Error ſhould ariſe in the right-hand Figure: however, as it may be ſafely uſed for 5 Minutes together, and ſometimes more, it will render the whole very eaſy.

Note, The Square of z in this Caſe, *viz.* the Arch of 5 Minutes, is ,00000211.

Thus having inveſtigated the *Newtonian* and our Author's Series, and exemplified the latter, by making the Sines of 30° 01′ and 30° 02′, and withal ſhewn how, from the Sine of an Arc given, to find the Length of that Arc, and conſequently the Circumference of the whole Circle; I ſhall beg leave, before I treat of the Conſtruction of Logarithms, to ſhew how, from the known Ratio of the Diameter to the Circumference, or any other Ratio whatſoever, that a Set of integral Numbers may be found, whoſe Ratio's ſhall be the neareſt poſſible to the Ratio given; for which I hope to be excus'd, and the rather, becauſe I believe this Method of determining them, was never before publiſh'd.

RULE.

Divide the Conſequent by the Antecedent, and the Diviſor by the Remainder, and the laſt Diviſor by the laſt Remainder, and ſo on till nothing remains.

Then for the Terms of the firſt Ratio, Unity will always be the Antecedent, and the firſt Quotient the firſt Conſequent.

For the TERMS of the ſecond RATIO:

Multiply the laſt $\left\{\begin{array}{l}\text{Antecedent}\\\text{Conſequent}\end{array}\right\}$ by the 2d Quotient, and to the Product add $\left\{\begin{array}{l}\text{Nothing;}\\\text{Unity;}\end{array}\right\}$ and ſo will the Reſult be the ſecond $\left\{\begin{array}{l}\text{Antecedent.}\\\text{Conſequent.}\end{array}\right.$

For all the following RATIO's:

Multiply the laſt $\left\{\begin{array}{l}\text{Antecedent}\\\text{Conſequent}\end{array}\right\}$ by the next Quotient, and to the Product add the laſt $\left\{\begin{array}{l}\text{Antecedent}\\\text{Conſequent}\end{array}\right\}$ but one; and ſo will the Sum be the preſent $\left\{\begin{array}{l}\text{Antecedent.}\\\text{Conſequent.}\end{array}\right.$

EXAMPLE.

Let it be required to find a Rank of Ratio's, whoſe Terms are integral, and the neareſt poſſible to the following Ratio, *viz.* of 10000 to 31416, which expreſſes nearly the Proportion of the Diameter of the Circle, to its Circumference.

But becauſe the Terms of the Ratio are not prime to each other, they muſt therefore be reduc'd to their leaſt Terms.

Whence $\frac{10000}{31418} = \frac{1250}{3927}$, and then 3927 divided by 1250 and 1250 by the Remainder, *&c.* will be as follows:

$$\begin{array}{lllll}
1250\ )\ 3927\ (\ 3 & & & & \\
 & 177\ )\ 1250\ (\ 7 & & & \\
 & & 11\ )\ 177\ (\ 16 & & \\
 & & & 1\ )\ 11\ (\ 11 & \\
 & & & \quad\ \ \underset{\smile}{0} &
\end{array}$$

So the firſt Antecedent is 1, and the firſt Conſequent 3.

$$\text{Laſt}\left\{\begin{array}{l}\text{Anteced. } 1\\ \text{Conſeq. } 3\end{array}\right\}\times 7 = \left\{\begin{array}{l}7\\ 21\end{array}\right\}\text{And}\left\{\begin{array}{l}7+0=7 \text{ the ſecond Anteced.}\\ 21+1=22 \text{ the ſecond Conſeq.}\end{array}\right.$$

Which 7 and 22 is *Archimedes*'s Proportion.

$$\text{Laſt}\left\{\begin{array}{l}\text{Anteced. } 7\\ \text{Conſeq. } 22\end{array}\right\}\times 16 = \left\{\begin{array}{l}112\\ 352\end{array}\right\}\text{And}\left\{\begin{array}{l}112+1=113 \text{ the 3d Anteced.}\\ 352+3=355 \text{ the 3d Conſeq.}\end{array}\right.$$

Which Terms 113 and 355 is *Metius*'s Proportion.

$$\text{Laſt}\left\{\begin{array}{l}\text{Anteced. } 113\\ \text{Conſeq. } 355\end{array}\right\}\times 11 = \left\{\begin{array}{l}1243\\ 3905\end{array}\right\}\text{And}\left\{\begin{array}{l}1243+7=1250\\ 3905+22=3927\end{array}\right.$$

Producing the ſame Antecedent and Conſequent as at firſt; which, as it is ever the Property of the Rule ſo to do, proves at the ſame time, that no Error has been committed thro' the whole Operation.

$$\text{Whence, as } 1250 : 3927 :: \left\{\begin{array}{r}1:3\\ 7:22\\ 113:355\end{array}\right\}\begin{array}{l}\text{For the}\\ \text{Terms}\\ \text{of the}\end{array}\left\{\begin{array}{c}1\\ 2\\ 3\end{array}\right\}\text{Ratio.}$$

But it muſt be obſerv'd, that 1 to 3 does not expreſs the Ratio ſo near as 7 to 22, nor 7 to 22 ſo near as 113 to 355; that is, the larger the Terms of the Ratio are, the nearer they approach the Ratio given.

Mr. *Molyneux*, in his Treatiſe of Dioptrics, informs us, that when Sir *Iſaac Newton* ſet about, by Experiments, to determine the Ratio of the Angle of Incidence, to the refracted Angle, by the means of their reſpective Sines; he found it to be from Air to Glaſs, as 300 to 193, or in the leaſt round Numbers, as 14 to 9. Now, if it be as 300 to 193, it will readily appear by the Rule, whether they are ſuch integral Numbers, whoſe Ratio is the neareſt poſſible to the given Ratio.

```
193 ) 300 ( 1
    107 ) 193 (
           86 ) 107 ( 1
                 21 ) 88 ( 4
                        2 ) 21 ( 10
                              1 ) 2 ( 2
                                  0
```

For dividing the greater Number by the leſs, and the leſs by the Remainder, *&c.* the Operation will ſhew, that the Numbers 193 and 300 are prime to each other; and that the firſt Antecedent is 1, as alſo the firſt Conqueſt.

$$\text{Whence}\left\{\begin{matrix}1\\1\end{matrix}\right\}\times 1=\left\{\begin{matrix}1\\1\end{matrix}\right. \text{And}\left\{\begin{matrix}1+0=1 & \text{the ſecond Antecedent.}\\1+1=2 & \text{the ſecond Conſequent.}\end{matrix}\right.$$

$$\text{Again}\left\{\begin{matrix}1\\2\end{matrix}\right\}\times 1=\left\{\begin{matrix}1\\2\end{matrix}\right. \text{And}\left\{\begin{matrix}1+1=2 & \text{the third Antecedent.}\\2+1=3 & \text{the third Conſequent}\end{matrix}\right.$$

$$\text{Again}\left\{\begin{matrix}2\\3\end{matrix}\right\}\times 4=\left\{\begin{matrix}8\\12\end{matrix}\right. \text{And}\left\{\begin{matrix}8+\quad=9 & \text{the fourth Antecedent}\\12+{}_2=14 & \text{the fourth Conſequent.}\end{matrix}\right.$$

Hence, the fourth Antecedent and Conſequent make the Ratio to be as 9 to 14, or inverſly as 14 to 9; which not only agrees with Mr. *Molyneux*, but at the ſame time diſcovers, that they are nearer to the given Ratio, than any other integral Numbers leſs than 92 and 143; which are the neareſt of all to the given Ratio, as will appear by repeating the Proceſs, according to the Direction of the Rule.

Sir *Iſaac Newton* himſelf determines the Ratio out of Air into Glaſs to be as 17 to 11; but then he ſpeaks of the Red Light. For that great Philoſopher, in his

Differ-

Diſſertations concerning Light and Colours, publiſh'd in the Philoſophical Tranſactions, has at large demonſtrated, as alſo in his Optics, that the Rays of Light are not all homogeneous, or of the ſame ſort, but of different Forms and Figures, ſo that ſome are more refracted than others, tho' they have the ſame or equal Inclinations on the Glaſs: Whence there can be no conſtant Proportion ſettled between the Sines of the Incidence, and of the refracted Angles.

But the Proportion that comes neareſt Truth, for the middle and ſtrong Rays of Light, it ſeems, is nearly as 300 to 193, or 14 to 9. In Light of other Colours the Sines have other Proportions. But the Difference is ſo little, that it need ſeldom to be regarded, and either of thoſe mention'd for the moſt part is ſufficient for Practice. However, I muſt obſerve, that the Notice here taken either of the one or the other, is more to illuſtrate the Rule, and ſhew, as Occaſion requires, how to expreſs any given Ratio in ſmaller Terms, and the neareſt poſſible, with more Eaſe and Certainty, than any Deſign in the leaſt of touching upon Optics.

Wherefore, leſt this ſmall Digreſſion from the Subject in hand, and indeed even from my firſt Intentions, ſhould tire the Reader's Patience, I ſhall not preſume more, but immediately proceed to the Conſtruction of Logarithms.

Of the Conſtruction of Logarithms.

THE Nature of which tho' our Author has ſufficiently explain'd in the Deſcription of the Logarithmical Curve; yet before we attempt their Conſtruction, it will be neceſſary to premiſe:

That the Logarithm of any Number is the Exponent or Value of the Ratio of Unity to that Number; wherein we conſider Ratio, quite different from that laid down in the fifth Definition of the 5th Book of theſe Elements; for beginning with the Ratio of Equality, we ſay 1 to 1=0, whereas, according to the ſaid Definition, the Ratio of 1 to 1=1; and conſequently the Ratio here mention'd is of a peculiar Nature, being affirmative when increaſing, as of Unity to a greater Number; but negative when decreaſing. And as the Value of the Ratio of Unity to any Number, is the Logarithm of the Ratio of Unity to that Number, ſo each Ratio is ſuppos'd to be meaſur'd by the Number of equal Ratiunculæ contain'd between the two Terms thereof; whence, if in a continued Scale of mean Proportionals, infinite in Number, there be aſſum'd an infinite Number of ſuch Ratiunculæ, between any two Terms in the ſaid Scale; then that infinite Number of Ratiunculæ is to another infinite Number of the like and equal Ratiunculæ between any other two Terms, as the Logarithm of the one Ratio is to the Logarithm of the other.

But if, inſtead of ſuppoſing the Logarithms compos'd of a Number of equal Ratiunculæ proportionable to each Ratio, we ſhall take the Ratio of Unity to any Number to conſiſt always of the ſame infinite Number of Ratiunculæ, their Magnitudes in this Caſe will be as their Number in the former. Wherefore, if between Unity, and any two Numbers propos'd, there be taken any Infinity of mean Proportionals, the infinitely little Augments or Decrements of the firſt of thoſe Means in each from Unity will be Ratiunculæ; that is, they will be the Fluxions of the Ratio of Unity to the ſaid Numbers; and becauſe the Number of Ratiunculæ in both are equal, their reſpective Sums, or whole Ratio's, will be to each other as their Moments or Fluxions; that is, the Logarithms of each Ratio will be as the Fluxion thereof. Conſequently, if the Root of any infinite Power be extracted out of any Number, the Difference of the ſaid Root from Unity ſhall be as the Logarithm of that Number. So that Logarithms thus produc'd may be of as many Forms as we pleaſe to aſſume infinite Indices of the Power whoſe Root we ſeek. As, if the Index be ſuppos'd 100000, *&c.* we ſhall have the Logarithms invented by *Napier*; but if the ſaid Index be 230258, *&c.* thoſe of Mr. *Briggs*'s will be produc'd.

Wherefore, if $1+x$ be any Number whatſoever, and n infinite, then its Logarithm will be as $\overline{1+x}^{n} - 1 = \frac{1}{n} \times x - \frac{xx}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$, *&c.* For the infinite Root of $1+x$ without its Unciæ, or prefixt Numbers, is $1+x+xx+xxx+xxxx$, *&c.* and the celebrated binomial Theorem invented by Sir *Iſaac Newton* for determining them is $1\times\frac{1}{n}\times\frac{\frac{1}{n}-1}{2}\times\frac{\frac{1}{n}-2}{3}\times\frac{\frac{1}{n}-3}{4}$, *&c.* or in this Caſe rather $1\times\frac{1}{n}\times-\frac{1}{2}\times\frac{-2}{3}\times\frac{-3}{4}\times\frac{-4}{5}$, *&c.* for $\frac{1}{n}$ being an infiniteſimal is rejected; whence the infinite Root of $1+x=\overline{1+x}^{\frac{1}{n}}=1+\frac{x}{x}-\frac{xx}{2n}+\frac{x^3}{3n}-\frac{x^4}{4n}+\frac{x_5}{5n}$, *&c.* and the Exceſs thereof above Unity, *viz.* $\frac{x}{n}-\frac{xn}{2n}+\frac{x^3}{3n}-\frac{x^4}{4n}$, *&c.* is the Augment of the firſt of the mean Proportionals between Unity and $1+x$, which therefore will be as the Logarithm of the Ratio of 1 to $1+x$, or as the Logarithm of $\overline{1+x}^{\frac{1}{n}}$. But as $1+x-1$ is a Ratiuncula, it muſt be multiplied by 10000, *&c.* infinitely, which will reduce it to Terms fit for Practice, making the Logarithm of the Ratio of 1 to $1+x=\frac{1000,\ \&c.}{n}\times\frac{x}{1}-\frac{xx}{2}+\frac{x^3}{3}-\frac{x^4}{4}+$, *&c.* whence if the Index n be taken 1000, *&c.* as in *Napier*'s Form, the Logarithms will be ſimply $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}$, *&c.* But as n may be taken at Pleaſure the ſeveral Scales of Logarithms to ſuch Indices will be as $\frac{1000,\ \&c.}{n}$, or reciprocally as their Indices.

Again, if the Logarithm of a decreaſing Ratio be ſought, the infinite Root of $1-x=\overline{1-x}^{\frac{1}{n}}$ will be found by the like Method to be $1-\frac{1}{n}x-\frac{1}{2n}x^2-\frac{1}{3n}x^3-\frac{1}{4n}x$, *&c.* which ſubtract from Unity, and the Decrement of the firſt of the infinite Number of Proportionals will appear to be $\frac{1}{n}\times x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}$

x^4, &c. which expreſſes the Logarithm of the Ratio of 1 to $1-x$, or the Logarithm of $1-x$ according to *Napier*'s Form, if the Index n be put $=10000$, &c. as before.

And to find the Logarithm of the Ratio of any two Terms, a the leaſt, and b the greater, it will be as $a : b :: 1 : 1+x : x = \frac{b}{a} - 1 = \frac{b-a}{a}$, or the Difference divided by the leſſer Term when 'tis an increaſing Ratio, and $\frac{b-a}{b}$ when 'tis decreaſing.

Wherefore putting $d=$ Difference between the two Terms a and b, the Logarithms of the ſame Ratio may be doubly expreſſed, and accordingly is either $\frac{1}{n} \times \frac{d}{a} - \frac{d^2}{2a^2} + \frac{d^3}{3a^3} - \frac{d^4}{4a}$, &c. or $\frac{1}{n} \times \frac{d}{b} + \frac{d^2}{2b^2} + \frac{d^3}{3b^3} + \frac{d^4}{4b^4}$, &c. both producing the ſame Thing.

But if the Ratio of a to b be ſuppoſed to be divided into two Parts, *viz.* into the Ratio of a to the arithmetical Mean between the two Terms, and the Ratio of the ſaid arithmetical Mean to the other Term b, then will the Sum of the Logarithms of thoſe two Ratio's be the Logarithm of the Ratio of a to b. Wherefore ſubſtituting $\frac{1}{2}s$ for $\frac{1}{2}a + \frac{1}{2}b$, and it will be $\frac{1}{2}s : a :: 1 : 1-x$; whence $x = \frac{\frac{1}{2}s - a}{\frac{1}{2}s} = \frac{d}{s}$. And again, as $\frac{1}{2}s : b :: 1 : 1+x$; whence $x = \frac{b - \frac{1}{2}s}{\frac{1}{2}s} = \frac{d}{s}$. Therefore ſubſtituting $\frac{d}{s}$ for x, and we ſhall have for both Ratio's $\frac{1}{n} \times \frac{d}{s} + \frac{d^2}{2s^2} + \frac{d^3}{3s^3} + \frac{d^4}{4s^4}$, &c. $\frac{1}{n} \times \frac{d}{s} - \frac{d^2}{2s^2} + \frac{d^3}{3s^3} - \frac{d^4}{4s^4}$, &c. And their Sum, *viz.* $\frac{1}{n} \times 2 \times \frac{d}{s} + \frac{d^3}{3s^3} + \frac{d^5}{5s^5} + \frac{d^7}{7s^7}$, &c. is the Logarithm of the Ratio of a to b, whoſe Difference is d, and Sum s; which Series without the Index n, is, by-the-bye, the

Fluent of the Fluxion of the Logarithm of $\frac{s+d}{s-d}$, aſſuming d the flowing Quantity, for the Fluxion of the Log. of $\frac{s+d}{s-d}$, is $\frac{2s\dot{d}}{ss--dd}=2\times\frac{\dot{d}}{s}+\frac{d^2\dot{d}}{s^3}+\frac{d^5\dot{d}}{s^5}+\frac{d^6\dot{d}}{s^7}$ +, &c. whoſe Fluent $2\times\frac{d}{s}+\frac{d^3}{3s^3}+\frac{d^5}{5s^5}+\frac{d^7}{7s^7}$, &c. is *Napier's* Logarithm of $\frac{s+d}{s-d}$ and the ſame as above, bating the Index n. This Series either way obtained converges twice as ſwift as the former, and conſequently is more proper for the Practice of making Logarithms; thus, put $a=1$, and b any Number at Pleaſure; then $\frac{d}{s}=\frac{b-1}{b+1}$, which aſſume $=e$, and then $b=\frac{e+1}{1-e}$ and becauſe $\frac{d}{s}=e$, therefore we have for

THEOREM I.

The Log. of b $\left(=\frac{e+1}{1-e}\right)=\frac{1}{n}\times 2\times e+\frac{1}{3}e^3+\frac{1}{5}e^5+\frac{1}{7}$, &c.

To illuſtrate this Theorem: Let it be required to find the Logarithm of 2 true to 7 Places.

Note, That the Index muſt be aſſumed of a Figure or two more than the intended Logarithm is to have.

EXAMPLE.

Here $b=\frac{e+1}{1-e}=2 \because e+1=2-2e \because 3e=1$; whence $e=\frac{1}{3}$, and $ee=\frac{1}{9}$.

The Operation ſtands thus:

$\frac{1}{3} = e =$	,33333333	$e =$	,33333333
$e^3 =$	3703704	$\frac{1}{3} e^3 =$	1234568
$e^5 =$	411523	$\frac{1}{5} e^5 =$	82304
$e^7 =$	45725	$\frac{1}{7} e^7 =$	6532
$e^9 =$	5080	$\frac{1}{9} e^9 =$	564
$e^{11} =$	564	$\frac{1}{11} e^{11} =$	51
$e^{13} =$	63	$\frac{1}{13} e^{13} =$	5
			,34657357
			2

Whence *Napier*'s Logarithm of 2 is ,69314714

But ,69314714 multiplied by 3 will give, 207944142 for the Logarithm of 8, inaſmuch as 8 is the Cube or third Power of 2, and the Logarithm of 8 plus the Log. of 1 $\frac{1}{4}$ is equal to the Logarithm of 10, becauſe 8 × 1 $\frac{1}{4}$ = 10; wherefore to find the Logarithm of 1 $\frac{1}{4}$ we have $b = \frac{e + 1}{1 - e} = \frac{1}{4} = \frac{5}{4}$; whence $e = \frac{1}{9}$ and $ee = \frac{1}{81}$.

The Operation ſtands thus:

$\frac{1}{9} = e =$	,11111111	$e =$	,11111111
$e^3 =$	137172	$\frac{1}{3} e^5 =$	45724
$e^5 =$	1693	$\frac{1}{4} e^5 =$	338
$e^7 =$	21	$\frac{1}{7} e^7 =$	3
			,11157176
			2

Whence *Napier*'s Logarithm of 1 $\frac{1}{4}$ is ,22314352
To which add the Logarithm of 8 2,07944142
The Sum, *viz.* 2,30258494

is *Napier*'s Logarithm of 10. But if the Logarithm of 10 be made 1,000000, *&c.* as it is moſt conveniently done in moſt of our Tables extant, then $\frac{2302585}{n} = 1{,}000$, *&c.* Whence $n = 2302585$, *&c.* is the Index for *Briggs*'s Scale of Logarithms; and if the above Work had been carried on to Places ſufficient, the Index n would have been 2,30258, 50929, 94045, 68401, 79914, 79914, *&c.* and its Reciprocal, *viz.* $\frac{1}{n} = 0,43429,$ 44819, 03251, 82765, 11289, *&c.* which, by the way, is the Subtangent of the Curve expreſſing *Briggs*'s Logarithms; from the Double of which the ſaid Logarithms may be had directly.

For, becauſe $\frac{1}{n} = 0,4342944$, *&c.* $\because \frac{2}{n} = ,868588 9638$, *&c.* which put $= m$, and then the Logarithm of $b = \frac{e+1}{1-e} = me + \frac{me^3}{3} + \frac{me^5}{5} + \frac{me^7}{7} + \frac{me^9}{9}$, *&c.*

EXAMPLE.

Let it be required to find *Briggs*'s Logarithm of 2.

Here $b = \frac{e+1}{1-e} = 2 \because e = \frac{1}{3}$ and $ee = \frac{1}{9}$.

The OPERATION ſtands thus:

$m = ,868588963$
$me = ,289529655$
$me^3 = 32169962$
$me^5 = 3574440$
$me^7 = 397160$
$me^9 = 44120$
$me^{11} = 4902$
$me^{13} = 545$

$me = ,289529655$
$\frac{1}{3}me^3 = 10723321$
$\frac{1}{5}me^5 = 714888$
$\frac{1}{7}me^7 = 56737$
$\frac{1}{9}me^9 = 4902$
$\frac{1}{11}me^{11} = 445$
$\frac{1}{13}me^{13} = 42$

Whence *Briggs*'s Logarithm of 2 is 0,30102999.

AGAIN:

Let it be requir'd to find *Briggs*'s Logarithm of 3; now becauſe the Logarithm of 3 is equal to the Logarithm of 2 plus the Logarithm of $1\frac{1}{2}$ (for $2 \times 1\frac{1}{2} = 3$) therefore find the Logarithm of $1\frac{1}{2}$, and add it to the Logarithm of 2 already found, the Sum will be the Logarithm of 3, which is better than finding the Logarithm of 3 by the Theorem directly, inaſmuch as it will not converge ſo faſt as the Logarithm of $1\frac{1}{2}$; for the ſmaller the Fraction repreſented by e, which is deduc'd from the No. whoſe Logarithm is ſought, the ſwifter does the Series converge.

Here $b = \frac{e+1}{1-e} = \frac{3}{2} \because 2e+2 = 3-3e \because e = \frac{1}{5}$, and $ee = \frac{1}{25}$.

The OPERATION is as follows:

$m =$	,868588963		
$me =$	,173717792	$me =$	,173717792
$me^3 =$	6948711	$\frac{1}{3}me^3 =$	2316237
$me^5 =$	277948	$\frac{1}{5}me^5 =$	55589
$me^7 =$	11117	$\frac{1}{7}me^7 =$	1588
$me^9 =$	447	$\frac{1}{9}me^9 =$	49
$me^{11} =$	17	$\frac{1}{11}me^{11} =$	1

Briggs's Logarithm of $1\frac{1}{2} =$,176091256
To which add the Logarithm of 2 = ,301029990
The Sum is the Logarithm of 3 = 0,477121246

Again, to find the Logarithm of 4, becauſe 2×2 = 4, therefore the Logarithm of 2 added to itſelf, or multiplied by 2, the Product 0,60205998 is the Logarithm of 4.

To find the Logarithm of 5, becauſe $\frac{}{2} = 5$, therefore from the Logarithm of 10 1,000000000
ſubtract the Logarithm of 2 ,301029990
There remains the Logarithm of 5 = ,698970010

And becauſe 2 × 3 = 6. Therefore

To find the Logarithm of 6

To the Logarithm of 3 ,477121246
Add the Logarithm of 2 ,301029990
The Sum will be the Logarithm of 6 = ,778151236

Which being known, the Logarithm of 7, the next prime Number, may be eaſily found by the Theorem; for becauſe $6 \times \frac{7}{6} = 7$, therefore to the Logarithm of 6 add the Logarithm of $\frac{7}{6}$, and the Sum will be the Logarithm of 7.

EXAMPLE.

Here $b = \frac{e+1}{1-e} = \frac{7}{6} \because e = \frac{1}{13}$ and $ee = \frac{1}{169}$.

$m = ,868588963$

$me = ,066814535$	$me = ,066814535$
$me^3 = 395352$	$\frac{1}{3}me^3 = -\,-\,131784$
$me^5 = 2339$	$\frac{1}{5}me^5 = 467$
$me^7 = 13$	$\frac{1}{7}me^7 = 2$

Briggs's Logarithm of $\frac{7}{6}$	,066946788
To which add the Log. of 6	,778151236
The Sum is the Log. of $7 =$	,845098024

Again, becauſe $4 \times 2 = 8$. Therefore

To the Logarithm of 4	,60205998
Add the Logarithm of 2	,30102999
The Sum is the Logarithm of 8	,90308997

And becauſe $3 \times 3 = 9$. Therefore

To the Logarithm of 3	,477121246
Add the Logarithm of 3	,477121246
The Sum is the Logarithm of 9	,954242492

And the Logarithm of 10 having been determined to be 1,0000000, we have therefore obtained the Logarithms of the firſt ten Numbers.

After the ſame manner the whole Table may be conſtructed, and as the prime Numbers increaſe, ſo fewer Terms of the Theorem are required to form their Logarithms; for in the common Tables, which extend but to ſeven Places, the firſt Term is ſufficient to produce the Logarithm of 101, which is compos'd of the Sum of the Logarithms of 100 and $\frac{101}{100}$, becauſe $100 \times \frac{101}{100} = 101$, in which Caſe $b = \frac{e+1}{1-e} = \frac{101}{100} \because e = \frac{1}{201}$, whence in making of Logarithms according to the preceding Method, it may be obſerv'd, that the Sum and Difference of the Numerator and Denominator of the Fraction whoſe Logarithm is ſought, is ever equal to the Numerator and Denominator of the Fraction repreſented by e; that is, the Sum is the Denominator, and the Difference, which is always Unity, is the Numerator; conſequently the Logarithm of any prime Number may be readily had by the Theorem, having the Logarithm either next above or below given.

Tho', if the Logarithms next above and below that Prime are both given, then its Logarithm will be obtained ſomething eaſier. For half the Difference of the

Ratio's which conſtitute the 1ſt Theorem, *viz.* ($n=$) $m \times \frac{dd}{2ss}+\frac{d^4}{4s^4}+\frac{d^6}{6s^6}+\frac{d^8}{8s^8}$, *&c.* is the Logarithm of the Ratio of the arithmetical Mean to the geometrical Mean, which being added to the half Sum of the Logarithms next above and below the Prime ſought, will give the Logarithm of that prime Number, which for Diſtinction's ſake may be called Theorem the ſecond, and is of good Diſpatch, as will appear hereafter by an Example.

But the beſt for this Purpoſe is the following one, which is likewiſe deriv'd from the ſame Ratio's as Theorem the firſt. For the Difference of the Terms between ab and $\frac{1}{4}ss$ or $\frac{1}{4}aa+\frac{1}{2}ab+\frac{1}{4}bb$, is $\frac{1}{4}aa-\frac{1}{2}ab+\frac{1}{4}bb=\overline{\frac{1}{2}a-\frac{1}{2}b}^2=\frac{1}{4}dd=$, 1, and the Sum of the Terms ab and $\frac{1}{4}ss$ being put $=y$, therefore (ſince y in this Caſe $=s$ and $d=1$) it follows that $\frac{1}{n}\times\frac{2}{y}+\frac{2}{y^2}+\frac{2}{y^5}+\frac{2}{y^7}$, *&c.* is the Logarithm of the Ratio of ab to $\frac{1}{4}zz$, whence $\frac{1}{n}\times\frac{1}{y}+\frac{1}{3y^3}+\frac{1}{5y^5}\frac{1}{7y}$, *&c.* is the Logarithm of the Ratio of $\frac{1}{2}s$ to $\sqrt{ab}$ which converges exceeding quick, and is of excellent Uſe for finding the Logarithm of prime Numbers, having the Logarithms of the Numbers next above and below given, as in Theorem the ſecond.

EXAMPLE.

Let it be required to find the Logarithms of the prime Number 101; then $a=100$ and $b=101$; whence $y=20401$, put $\frac{1}{n}=m=,4342944819$, *&c.* then the Series will ſtand thus; $\frac{m}{y}+\frac{m}{3y^3}+\frac{m}{5y^5}+\frac{m}{7y^7}$, *&c.*

And

And $m=$,43429, *&c.* divided by $y=20401$, quotes	,0000212879014
Therefore to the half Sum of the Logarithms of 100 and 102 =	2,0043000858809
Add the ſaid Quote	0,20000212879014
And the Sum, *viz.*	2,0043213737823

is the Logarithm of 101 true to 12 Places of Figures, and obtained by the firſt Term of the Series only; whence 'tis eaſy to perceive what a vaſt Advantage the ſecond Term would have, were it put in Practice, ſince m is to be divided by 3 multiplied into the Cube of 20401.

This Theorem which we'll call Theorem the third, was firſt found out by Dr. *Halley*, and a notable Inſtance of its Uſe given by him in the Philoſophical Tranſactions for making the Logarithm of 23 to 32 Places, by five Diviſions performed with ſmall Diviſors; which could not be obtained according to the Methods firſt made uſe of, without indefatigable Pains and Labour, if at all; on account of the great Difficulty that would attend the managing ſuch large Numbers.

Our Author's Series for this Purpoſe is (*Page* 363) $y\times\frac{1}{4z}+\frac{1}{24z^3}+\frac{7}{360z^5}$, *&c.* the Inveſtigation of which as he was pleas'd to conceal, induc'd me to inquire into it, as well to know the Truth of the Series, as to know whether this or that had the Advantage, becauſe Dr. *Halley* informs us, when his was firſt publiſhed, that it converged quicker than any Theorem then made public, and in all Probability does ſo ſtill; however that be, 'tis certain our Author's converges no faſter than the ſecond Theorem, as I found by the Inveſtigation thereof, which may be as follows:

From the foregoing Doctrine the Difference of the Logarithms of $z-1$ and $z+1$ is $m\times\frac{2}{z}+\frac{2}{3z^3}+\frac{2}{5z^5}+\frac{2}{7z^7}+\frac{2}{9z^9}$, *&c.* which put equal to y, and the Logarithm of the Ratio of the Arithmetical Mean z, to the Geometrical Mean $\sqrt{zz-1}$ is $m\times\frac{1}{2zz}+$ $+\frac{1}{4z^4}+\frac{1}{6z^6}+\frac{1}{8z^8}$, *&c.* *per* Theorem the ſecond, for z equal to $\frac{1}{2}s$; whence $\frac{dd}{2ss} = \frac{1}{2zz}$.

Let A and B be the Logarithms of $z - 1$ and $z + 1$ reſpectively; then is $\frac{A+B}{2} + m \times \frac{1}{2zz} + \frac{1}{4z^4} + \frac{1}{6z^6} + \frac{1}{8z^8}$, *&c.* the Logarithm of z, and if the latter Part of the Series expreſſing the ſaid Logarithm of z be divided by the Series repreſenting the Difference of the Logarithms of $z - 1$ and $z + 1$, the Quotient will exhibit the Series required, *viz.* $\frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z^5}$ *&c.* as appears by the following Operation:

$$\frac{2}{z} + \frac{2}{3z^3} + \frac{2}{5z^5}\,\&c.\Big)\,\frac{1}{2z^2} + \frac{1}{4z^4} + \frac{1}{6z^6}\,\&c.\Big(\frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z^5}\,\&c.$$

$$\underline{\frac{1}{2z^2} + \frac{1}{6z^4} + \frac{1}{10z^6},\ \&c.}$$

$$\frac{1}{12z^4} + \frac{1}{15z^6},\ \&c.$$

$$\underline{\frac{1}{12z^4} + \frac{1}{36z^6},\ \&c.}$$

$$\frac{1}{180z^6},\ \&c.$$

Now, becauſe the Dividend is ever equal to the Diviſor drawn into the Quotient of the Diviſion, it follows that $y \times \overline{\frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z^5}}$, *&c.* is equal to $m \times \overline{\frac{1}{2zz} + \frac{1}{4z^4} + \frac{1}{6z^5}}$, *&c.* But $\frac{A+B}{2} + m \times \frac{1}{2zz} + \frac{1}{4z^4} + \frac{2}{6z^6}$, *&c.* is the Logarithm of z. Wherefore $\frac{A+B}{2} + y \times \frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z^5}$, *&c.* is the Logarithm of z. Q. E. I.

Note, I make the Author's 5th Term $\frac{13}{25200z^9}$ to be $\frac{1903}{226800z^9}$.

To illuſtrate this Theorem by an Example:
Let it be required to find the Logarithm of 101.
To the half Sum of the Logarithms of 101 and 102 = 2,0043000858
Add the Difference of the ſaid Logarithms divided by 4 z equal to } 0,0000212875
And the Sum, *viz.* 2,0043213733
is the Logarithm of 101 true to 9 Places of Figures; whence it appears, that our Author's Series falls ſhort of Dr. *Halley*'s in finding the Logarithm of the prime Number 101, three Places of Figures, by uſing only the firſt Terms of the Series; whereas, if two Terms in each were uſed, perhaps the Difference would have been conſiderably greater.

Note, This Series of our Author, deduced from Theorem the Second, is in Effect Dr. *Halley*'s too, but veil'd over by being thrown into a different Form: which however has its Uſe, as being very ready in Practice.

Having thus inveſtigated ſeveral Theorems, whereby the Tables of Logarithms, according to any Form, may be conſtructed, it remains to ſhew how from the Logarithm given to find what Ratio it expreſſes.

The Logarithm of the Ratio of 1 to $1+x$ has been prov'd to be as $\overline{1+x}^{\frac{1}{n}}-1=\frac{1}{n}\times x-\frac{1}{2}x_2+\frac{1}{3}x^3-\frac{1}{4}x^4$, *&c.* n being any infinite Index whatſoever; whence if L be put for the ſaid Series, then $\overline{1+x}^{\frac{1}{n}}-1=L$; conſequently $\overline{1+x}^{\frac{1}{n}}=1+L$, and $1+x=\overline{1+L}^{n}=1+nL+\frac{1}{2}n^2L^2+\frac{1}{6}n^3L^3+\frac{1}{24}n^4L^4$, *&c.*

A G A I N:

The Logarithm of the Ratio of 1 to $1-x$ has likewiſe been prov'd to be as $1-\overline{1-x}^{\frac{1}{n}}=\frac{1}{n}\times x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4$, *&c.* $=L$; wherefore $\overline{1-x}^{1}=1-L$, and $1 - x = \overline{1 - L}^{\,n} = 1 - nL + \frac{1}{2}n^2L^2 - \frac{1}{6}n^3L^3 + \frac{1}{24}n^4L^2$, &c.

Whence $1 \pm x = 1 \pm nL + \frac{1}{2}n^2L^2 \pm \frac{1}{6}n^3L^3 + \frac{1}{24}n^4L^4$, &c. is a general Theorem for finding the Number from the Logarithm given of any Species or Form whatſoever; but in the Application of it to Practice we labour under a great Inconveniency, eſpecially if the Numbers are large; that is to ſay, it converges ſo very ſlow, that it were much to be wiſhed it could be contracted.

However, if L be the Logarithm of the Ratio of a the leſſer Term, to b the greater, and either of them are given; then the other will be eaſily had and expeditiouſly enough too:

For $\left.\begin{array}{l} b \\ a \end{array}\right\} = \left\{\begin{array}{l} a \\ b \end{array}\right\} \times 1 \pm nL + \frac{1}{2}n^2L^2 \pm \frac{1}{6}n^3L^3$, &c.
and

Wherefore it follows by the Help of a Table of Logarithms, that the correſponding Number to any Logarithm may be found to as many Places of Figures as thoſe Logarithms conſiſt of; for putting d equal to the Difference between the given Logarithm and the next leſs in the Table, then will the Number ſought, *viz.* $N = a \times 1 + nd + \frac{1}{2}n^2d^2 + \frac{1}{6}n^3d^3$ &c. But if d be put equal to the Difference between the given Logarithm, and the next greater, then $N = b \times 1 - nd + \frac{1}{2}n^2d^2 - \frac{1}{6}n^3d^3$, &c. Both of which Series converge faſter, as d is ſmaller.

But the firſt three Terms in each may be contracted into two, which is very uſeful, inaſmuch as it ſaves the Trouble of raiſing n and d in the third Term to the ſecond Power; for letting the firſt Term remain as it is, the other two are reduced to one; thus, make the ſecond Term the Numerator of a Fraction, and Unity minus the third Term divided by the Second is the Denominator.

Whence $N = a \times 1 + nd + \frac{1}{2}n^2d^2 = a + \dfrac{ad}{\frac{1}{n} - \frac{1}{2}d}$ and $N = b \times d^2 - nd + \frac{1}{2}n^2 = b - \dfrac{bd}{\frac{1}{n} + \frac{1}{2}d}$.

Wherefore $a + \dfrac{ad}{m - \frac{1}{2}d}$, or $b - \dfrac{bd}{m + \frac{1}{2}d}$ will be the

Number anſwering to the given Logarithm, which tho' it differs a little from the Truth, is ſufficient to find the Numbers exact to as many Places as *Briggs*'s Logarithms conſiſt of, *viz.* 14, which are the largeſt Tables extant. Much after the ſame Method may the whole Series be contracted, by which means each alternate Power of d will be exterminated, or, which is the ſame thing, every two Terms in the Series will be reduc'd to one, making the whole as ſhort again.

To illuſtrate theſe Contractions by an Example:

Let it be required to find the Number anſwering to the Logarithm 7,5713740282 in *Briggs*'s Form.

From the given Logarithm	7,5713740282
Subtract the Log. of 37271000 the next neareſt	7,5713710453
the Remainder is equal to $d=$	,0000029829

And becauſe the Number 37271000 is leſs than the Number ſought, call it a, which, multiplied by ,0000029829, and the Product 1,11175 6659, &c. divided by $m-\frac{1}{2}d=$,434299, &c. quotes ,255,992, which, added to 37271000, gives 37271255,992 for the Number ſought.

Thus, I preſume, the Doctrine of Logarithms has been ſufficiently exemplified, whether we conſider the Conſtruction of them for any given Numbers, or on the contrary the finding of the Numbers from the Logarithms given.

But before I conclude, I ſhall give an Inſtance or two of the great Uſe of Logarithms in Arithmetical Calculations, and firſt in the purchaſing of Annuities.

If a be put for any Annuity, p for the preſent Value, r the Amount of one Pound for one Year at any Rate of Intereſt, and t for the Time or Number of Years the Annuity is to continue, then, $p=\frac{a-\frac{a}{r^t}}{r-1}$ the Value of the Annuity.

EXAMPLE.

Let it be required to find the preſent Value of an Annuity of 60 *l. per Annum* to continue 75 Years at the Rate of 4 *per Cent. per Annum.*

Here $a = 60$, $t = 75$, and $r = 1{,}04$. Now, in order to obtain the Anſwer we muſt find the ſeventy-fifth Power of r, or of 1,04; that is, we muſt multiply 1,04 ſeventy-five times into itſelf, which is exceeding tedious by the common way, as any one may judge; but by the Logarithms 'tis done with the greateſt Eaſe; for if 0,017033 the Logarithm of 1,04 be multiplied by 75, the Product 1,277475 will be the Logarithm of the ſeventy-fifth Power of 1,04; which being ſubtracted from 1,778151, the Logarithm of a equal to 60, will leave 0,500676 the Logarithm of 3,16719, which being ſubtracted from 60, and the Remainder divided by $r - 1 = {,}04$ will give 1421,820 equal to 1421 *l.* 15 *s.* 4 *d.* $\frac{3}{4}$ for the Value of the Annuity; and if 1421,820 be divided by 60, the Quotient will exhibit the Number of Years Purchaſe requiſite to be given for any Annuity to continue 75 Years upon a good Security free of all Incumbrances, the Purchaſe being made at 4 *per Cent.*

Hence we ſee the Reaſon why the long Annuities purchaſed in the Year 1708, having about 75 Years to come, are now valued in *Caſtain*'s Bill of Exchanges at 24 $\frac{1}{2}$ or 25 Years Purchaſe; for tho', according to this Calculation, they are worth but a little more than 23 Years and a half, yet becauſe in the public Funds 4 *per Cent.* is ſcarcely ever made of Money, and the Contingencies it is there ſubject to, which thoſe Annuities and other Government Securities are not, makes them very juſtly worth 24 $\frac{1}{2}$ or 25 Years Purchaſe.

Likewiſe Queſtions relating to Annuities upon Lives, whether for one, two, or three, *&c.* are almoſt as eaſily eſtimated. For Inſtance, it may readily be found by Logarithms, that an Annuity for a Man of Thirty to continue during his Life is worth 11,61 Years Purchaſe, Intereſt 6 *per Cent.* but at 4 *per Cent.* 14,68. And as the Probabilities of Life's Continuance, and the Value thereof, are determined by an Algebraical Proceſs grounded upon the Rudiments of the Doctrine of Chances, and five Years Obſervations upon the Bills of Mortality of *Breſlaw*, the Capital of *Sileſia*; ſo there reſults that Truth and Equity from the Operations, as ought to preſide in all Contracts of this Nature. Whence it follows, that all other Methods, whoſe Reſolution differs from this, (eſpecially if the Difference be much) may juſtly be deem'd erroneous, conſequently prejudicial to one of the Parties concerned. Wherefore, to prevent Impoſitions thro' Ignorance, great Care ſhould be taken; which Precaution, however unneceſſary it may appear, 'tis preſum'd, will be regarded, inaſmuch as no one is willing to pay more Years Purchaſe than he has Chances for living, as on the contrary the Seller to receive leſs than his Due, which may poſſibly be by following the common Methods, where, for the moſt part, regard is had neither to Age nor Intereſt, but founded upon Caprice, Humour, or, if you pleaſe, Cuſtom, the Contract being made, as they can agree, right or wrong; which Method of Procedure ought to be exploded, ſince ſo liable to Error, and the Conſequences drawn therefrom, ſo often wide of the Truth.

The other Inſtance which I ſhall give of the great Uſe of Logarithms is in the Caſe of *Seſſa*, as related by Dr. *Wallis* in his *Opus Arithmeticum* from *Alſephad* (an *Arabian* Writer) in his Commentaries upon *Tograius*'s Verſes, namely that one *Seſſa* an *Indian* having firſt found out the Game at *Cheſſe*, and ſhewed it to his Prince *Shehram*, the King, who was highly pleas'd with it, bid him ask what he would for the Reward of his Invention; whereupon he ask'd, That for the firſt little Square of the *Cheſſe* Board, he might have one Grain of Wheat given, for the ſecond two, and ſo on, doubling continually, according to the Number of the Squares in the *Cheſſe* Board, which was 64. And when the King, who intended to give a very noble Reward, was much diſpleas'd, that he had ask'd ſo trifling a one, *Seſſa* declar'd, that he would be contented with this ſmall one. So the Reward he had fix'd upon was order'd to be given him: But the King was quickly aſtoniſh'd, when he found, that this would riſe to ſo vaſt a Quantity, that the whole Earth itſelf could not furniſh out ſo much Wheat. But how great the Number of theſe Grains is, may be found by doubling one continually 63 times, ſo that we may get the Number that comes in the laſt Place, and then one time more to have the Sum of all: For the Double of the laſt Term leſs by one is the Sum of all. Now this will be more expeditiouſly done by Logarithms, and accurately enough too for this Purpoſe. For if to the Logarithm of 1, which is 0, we add the Logarithm of 2, which is

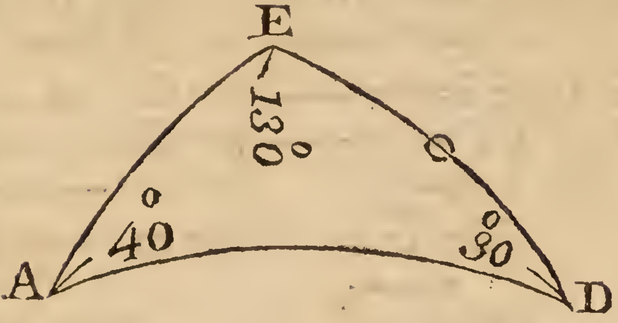
0,3010300 multiplied by 64, that is, 19,2659200, the abſolute Number agreeing to this will be greater than 18466, 00000, 00000, 00000, and leſs than 18447, 00000, 00000, 00000.

As I have had the reviſing of theſe Sheets, ſo it may be expected, that I ſhould give my Opinion concerning Mr. *Cunn* and our Author, in regard to ſpherical Trigonometry; wherein the former accuſes the latter, and ſeveral other eminent Authors, of having committed many Faults, and in ſome Caſes of being miſtaken, eſpecially in the Solution of the 12*th* Caſe of Oblique Spherics; in which Mr. *Cunn* has intirely miſtook the Author's Meaning, as plainly appears by his Remark, where he conſtitutes a Triangle, whoſe Sides are equal to the given Angles; whereas the Author means, that each Angle ſhould firſt be chang'd into its Supplement, and then with the ſaid Supplement another Triangle conſtituted, whoſe Angles by the very Text of the 14*th* Propoſition of his own Spherical Trigonometry, will be the Supplements of the Sides ſought in the given Triangle; to which Propoſition I refer the Reader. That this is the Senſe of the Author, is very evident, if impartially attended to, and which I think could poſſibly have no other Meaning; and accordingly aver what is here advanc'd to be univerſally true; but becauſe I would not be miſunderſtood, ſhall illuſtrate the Truth thereof by a numerical Operation; which, to thoſe who care not to trouble themſelves with the Demonſtration, may be ſufficient, and to others, ſome Satisfaction.

EXAMPLE.

Suppoſe, in the Oblique-angled Spherical Triangle ADE, there are given the Angles A, D, E, as *per* Figure, and the Side DE required.

Note, Write down the Supplements of the two Angles next the Side required firſt; and then the Operation may ſtand thus;



The Supplement of the Angle.	E = 50°	Sine Co. Ar.	0,115746
	D = 150	Sine Co. Ar.	0,301030
	A = 140	Sine 120°	9,937530
	Sum = 340	Sine 20	9,535052
	2|		
	½ Sum = 170		19,884358
			2|
½ Sum, minus the Supplement of the Angle.	E = 120		
	D = 20		9,944179

Which laſt Figures, 9,944179, give the Sine of 61° 34′; and the Double thereof, *viz.* 123° 08′, ſubtracted from 180 Degrees, leaves for the Supplement 56° 52′, which is the Side D E required.

The Rule which Mr. *Cunn* ſubſtitutes in the room of our Author's, is alſo univerſal (but not new); and conſequently, when he ſays, Change one of the Angles adjacent to the Side ſought into its Supplement, it is very juſt: tho' by the way I affirm, it is equally true, if the Angle oppoſite to the Side ſought were changed into its Supplement (which perhaps is what has not yet been taken Notice of); only then, inſtead of having the Side ſought directly, we ſhould have its Complement to 180 Degrees, as in the precedent Example; but there is a Neceſſity of changing either one or all of the Angles into their Supplements, tho' it is beſt to change only one; which let be either of thoſe next the Side ſought, no matter which; and the Side will be had directly without any Subduction, as will appear by the ſubſequent Operation.

EXAMPLE.

Let the Angle E be changed into its Supplement, and the Side D E ſought; which Supplement, and the other Angle adjacent to the Side ſought, being written down firſt; the Operation may be as follows:

Sup. of the Angle E=50	Sine Co. Ar	0,115746
The Angle { D=30	Sine Co. Ar	0,301030
A=40	Sine 10°	9,239670
Sum 120	Sine 30	9,698970
2|		
½ Sum 60	Sum	19,355416
½ Sum minus { Sup. Angle E=10 Angle D=30	½ Sum	9,677708

Which half Sum 9,677708 gives the Sine of 28° 56′, and the Double thereof 56° 52′ is the Side D E ſought, the ſame as before, when all the Angles were chang'd into their Supplements.

Whence it is abundantly manifeſt, that thoſe two Methods of Operation, notwithſtanding their Manner is ſo different, agree preciſely in Practice; and conſequently we may conclude our Author's Rule to be right. Wherefore I wonder Mr. *Cunn* did not attend better to the Words of our Author's Rule, before he ventur'd to attack the Characters of ſo many famous Trigonometrical Writers. But to remove the Imputation of his Charge againſt thoſe Authors who have deſerv'd ſo well of the Mathematics, and to juſtify them to the World, (for Juſtice ought to have place) it is, that I have ventur'd to give my Opinion, and point out where Mr. *Cunn* was miſtaken: The Reaſon of which is not eaſily aſſign'd, ſince, to give him his Due, it could not be for want of Knowledge, though in this Caſe I can't think it intirely owing to Inadvertency, inaſmuch as it was a premeditated Thing; and I am loth to impute it to any contentious Inclinations of his, in diſputing the Veracity of our Author's Rule, becauſe it did not appear with all that Plainneſs requiſite to prevent carping by the Litigious; wherefore, as I am in Suſpenſe how to determine, ſhall leave the Deciſion thereof to better Judgments.

Indeed, Mr. *Heynes*'s Rule, which directs with the three Angles given to project a Triangle, as if they were Sides, is deficient, were it only on that very account; for with the given Angles in the preceding

Example, it will be impoſſible to conſtruct a Triangle, becauſe 'tis requiſite, that two Sides together, however taken, be greater than the third, whereas in this Caſe they will be leſs: But the Rule is not only deficient in that reſpect, but really wrong; for though what Mr. *Heynes* aſſerts it juſt, *viz.* that the greateſt Side in the ſupplemental Triangle is the Supplement of the greateſt Angle in the other Triangle; yet, notwithſtanding that, the Conſequence drawn therefrom is falſe, and ſo the Solution only imaginary: For with Submiſſion, neither the Sides, nor their Supplements, in Mr. *Heynes*'s ſupplemental Triangle, are the Meaſures of the Sides ſought. 'Tis true, when one of the Angles is a right one, and the others both acute, then the ſaid Supplemental Triangle is that wanted to be conſtructed, as containing all the given Angles; and conſequently the Sides appertaining thereto, are the very Sides required: But then this is only one Inſtance out of the infinite Number of other Triangles that may be conſtructed; and which is not ſolved directly by the Triangle firſt projected neither; for the greateſt Angle thereof muſt be changed into its Supplement, when the Side oppoſite to the right Angle is required; and if the right Angle ſtill remains, and either one or both of the other given Angles are obtuſe, the Solution is rendered more perplex'd: Wherefore there can be no general Solution given to any Triangle, by conſtituting a Triangle, whoſe Sides are equal to the given Angles; except to that particular one which Mr. *Cunn* takes Notice of in his Remark, where each given Angle is the Meaſure of its oppoſite Side ſought, and which therefore needs no Operation.

This I thought myſelf obliged to obſerve, out of Juſtice to Mr. *Cunn*, who we ſee is not intirely to blame; as having juſt Reaſon to object againſt the Veracity of Mr. *Heynes*'s Rule, tho' not againſt the Rules of the other Authors by him nominated.

And here I can't but take Notice of ſome Gentlemen, who are ſo very fond of finding Fault, that rather than you ſhall not be in the wrong, they will wreſt your own Meaning from you, and will not ſuffer an Error, tho' ever ſo minute, to paſs, without proclaiming it to the Public, under Pretence of preventing their being impoſed upon; whereas, if the Truth were known, I fear it would appear to be the Vanity of their Hearts, an Over-fondneſs of being thought wiſer and more knowing than the reſt of Mankind; nay, I think it appears plainly ſo, by their oppoſing the Works of Men greater than themſelves: But if, inſtead of comparing how far their finite Knowledge extended or exceeded another Perſon's, they conſider'd how much there was they knew nothing of; as it would conduce to make them humble, ſo I am of Opinion it would contribute very much toward their leaving off that manner of Writing. Beſides, as I take it, the Buſineſs of Writing is not ſo much to diſcover who has committed the moſt Faults, as to avoid them, and make greater Improvements.

But what is the moſt to be wonder'd at, thoſe who are ſo very ready in finding fault, not without great Suſpicion, receive the beſt part of their Knowledge from the Works of thoſe very Authors againſt whom they exclaim. The Reaſon that induces me to think ſo, is this: Whilſt they are ſtudying an Author in order to underſtand him, then it is perhaps, they diſcover ſomething which he was pleas'd to omit, or thought fit to conceal, for which 'tis more than probable they take care not to omit paying a profound Reſpect to their vainly imagin'd ſuperior Genius's: And if by Accident an Error ſhould creep in, (which is very poſſible, none being infallible) then to be ſure he muſt be egregiouſly miſtaken, and not underſtand what he was about: But, I ſay, this Diſquiſition into the Demerits of an Author would never have been made, had they underſtood the Subject beforehand; for if otherwiſe, they muſt be of a ſad Cynical Temper, as well as have little elſe to do,to make it their Buſineſs to diſcover Faults, and at the ſame time acknowledge not one ſingle Beauty; a very ingrateful Return for the Advantage they receive in the Peruſal.

Nor do they do the Public that Service they pretend to; for thoſe that are capable, and will be at the Trouble, of reading a Treatiſe upon a Subject without a Maſter, are as well able as themſelves to rectify what is amiſs; and as for thoſe who will not be at that Trouble, there is no Danger of their being led aſtray; ſince it is the ſame thing to them, whether there be any Miſtakes or not.

However, if, after all, there ſhould be a Neceſſity for an Admonition, why can't it be done with Candour and

Humanity? And then without Doubt, an Author, out of regard to Truth, which of all things ought to be preferr'd, would be thankful: And to reprove otherwiſe, is to be ungenerous; becauſe whenever thoſe Miſtakes happen, as they are for the moſt part owing more to Inadvertency, than want of Knowledge; ſo they ſhould therefore be attributed to the Frailty of human Nature, (to which we are all more or leſs ſubject) nothing being more common amongſt all Profeſſions, than the writing of one thing for another.

If any think, by my interfering between our Author and Mr. *Cunn*, that I have run into the ſame Error, of which I accuſe others in general of being guilty, let them pleaſe to conſider, that I have only writ in the Vindication of Gentlemen, who were firſt wrongfully accus'd; and in one Particular juſtify'd Mr *Cunn*; for ſuch an Occaſion as this offering, I thought the Difference between them lay upon me to decide; leſt I ſhould be taxed with Partiality for not doing Juſtice, or with Ignorance in not determining an Affair which held ſome in Suſpenſe to know who was in the right or wrong; for there could be no Poſſibility of making a Merit, in adjuſting a thing of ſo eaſy a Nature; tho' perhaps, to conceive thoroughly the Reaſon of all the different Methods of Solution, may not be ſo eaſy neither.

But to proceed, as for the Omiſſions our Author has made in not determining accurately, when ſome of the Caſes are ambiguous, and when not, I ſhall not quarrel with thoſe who think him to blame; but if I may be allowed to give my Opinion, I think they are determined for the moſt part as well, or at leaſt with more Eaſe, from the Conſtruction of the Triangles, becauſe it fixes an Idea of what one is about, by exhibiting a kind of an ocular Demonſtration; and conſequently, prevents the laying of that Streſs upon the Memory, as all thoſe are obliged to, who depend intirely upon Mr. *Cunn*'s Rules; which to Beginners is not very agreeable: Hence who knows, but that what our Author wrote relating to the ambiguous Caſes, he thought ſufficient? that is, that the Reader would not ſtop, for want of farther Explications, but with more Eaſe ſupply himſelf with what was wanting, when he came to the Practice thereof, I mean, the Conſtruction of Triangles (for after all, without the Knowledge of that, a Perſon will have but a mean Notion of this uſeful Branch of the Mathematics); and if ſo, he ought in ſome meaſure to be excuſed, eſpecially if to this we join the following Conſideration, *viz.* that few or none ever learn Spherical Trigonometry, purely for the ſake of calculating Sides and Angles, to determine their Ambiguities; beſides, what is ambiguous in Trigonometry, is very often not ſo in Geography and Aſtronomy, *&c.* for which the other is chiefly learnt.

For Inſtance; If we know the Latitude of *London*, and the Diſtance and Difference of Longitude between the ſaid Place and *Rome*, notwithſtanding there are two Sides and an Angle oppoſite to one of them given, the Caſe is not doubtful when we undertake to find the Latitude of *Rome*; unleſs it be not known whether it lies to the *Northward* or *Southward* of *London*; which however could not be determined by any Principles of Trigonometry. Likewiſe in Aſtronomy, if the Latitude of the Place, the Sun's Declination and Azimuth were given, the *Quæſitum* is not doubtful neither, unleſs the Sun's Declination exceeds the Latitude of the Place, and both of the ſame Denomination, that is, both *North*, or both *South*; in which Caſes becauſe it is poſſible for the Sun to be upon the ſame Azimuth Circle, twice in the Forenoon; and upon another Azimuth Circle, twice in the Afternoon; it is doubtful, if by Circumſtances during the Obſervation, we can't diſcover which of the Times, whether the firſt or laſt; but if thoſe Times fall near each other, it will be quite impoſſible to diſtinguiſh which, and therefore ambiguous. Other Inſtances might be produced, but I believe theſe are ſufficient, to evince that thoſe nice Diſtinctions are not ſo neceſſary in Practice; if there be thoſe who think otherwiſe, I ſhall not diſpute it, but leave them to their Opinion without Interruption.

However, what with Mr. *Cunn*'s Rules for determining the ambiguous Caſes, (which are judiciouſly drawn up, as including all the Varieties poſſible) and the Corrections now made by reſtoring what was loſt and corrupted, our Author's Treatiſe of Trigonometry, in reſpect to Theory, may perhaps appear complete, even to the moſt ſcrupulous. And

Here I thought to conclude; but for the ſake of Novelty, and to illuſtrate the various Methods for ſolving

ing the 12*th* Caſe of Oblique Spherics, where the three Angles are given to find either of the Sides, I ſhall beg leave to give one Inſtance more, in order to ſhew how it may be perform'd after a new manner, by the Help of the natural and logarithmical verſed Sines; which if not intirely new, is not ſo publicly known as the preceding Methods; at leaſt I never ſaw any-where the Method of Operation, and therefore ſhall deliver a Rule for that Purpoſe in the following Words:

RULE.

Having, according to the former Directions, chang'd one of the Angles next the Side ſought into its Supplement; take the natural verſed Sine of the Difference of the ſaid Supplement and the other adjacent Angle, and ſubtract it from the natural verſed Sine of the Angle oppoſite to the Side ſought, and to the Logarithm of the Remainder add the Square of the Radius; then from the Sum ſubtract the logarithmical Sines of the above Supplement, and the ſame adjacent Angle; and the Remainder is the Logarithm of a Number, which will be the verſed Sine of the Side ſought.

EXAMPLE.

Supplement <E — 50°
Angle D —— 30
Natural { Diff. — 20 — ,06030
V. Sine { <A — 40 — ,23395
,17365
The Log. of which Diff. 17365 } 29,239674
with the Square of Radius is }
Sine of the Sup. of the <E 50° — 9,884254
Add the Sine of the <D 30 — 9,698970
Sum ſubtract 19,583224
Remains 9,656450

Which Remainder 9,656450 gives the Logarithm verſed Sine of DE 56° 52′, agreeing exactly with the former Computations.

Note, If the ſaid Remainder exceeds 10,000000, it implies that the Side ſought is greater than a Quadrant; wherefore cancelling the Characteriſtic 10, look out for the Number anſwering the remaining Logarithm, from which cut off the left-hand Figure, or, which is the ſame thing, abate the Radius (*viz.* Unity); and the Remainder will be the natural Sine of the Exceſs of the Side ſought above a Quadrant.

As the natural and logarithmical verſed Sines are not ſo frequently met with in Books, as the artificial Sines, 'tis poſſible, on that account, this Rule may meet with ſome Objection; for which Reaſon, and not knowing whether it may be thought preferable to the foregoing Methods, (tho' undoubtedly very eaſy in Practice) I have omitted its Demonſtration; but have publiſh'd the Rule, with ſome View of introducing the Uſe of the former Sines, which ſometimes are preferable to the latter: For by the Help of the ſaid verſed Sines, and the Reaſoning us'd in obtaining this Rule, we neceſſarily come to the Knowledge of ſolving that Problem, where two Sides and the contained Angle are given, and the third Side required, at one Operation, very uſeful in Aſtronomy and Geography, eſpecially in the latter; when the Latitudes and Longitudes of two Places are given to find their Diſtance aſunder: But the Rule for performing it, and the Demonſtration thereof, is alſo omitted for the ſake of Brevity.

However, 'tis eaſy to perceive, ſince Angles may be turned into Sides, that the preſent Rule includes the Solution of that uſeful Problem in Aſtronomy, for finding the Sun's Azimuth, having the Latitude of the Place, the Sun's Altitude and Diſtance from the elevated Pole given; by which means the Variation of the Compaſs of ſuch Importance to Navigators, may be readily determined in any Part of the World.

An Example of which, comprehending the latter Part of the Rule, *viz.* (when the Remainder exceeds 10,000000) is exhibited.

EXAMPLE.

Suppoſe on *June* the 30*th* 1732. at *London* in the Latitude of 51° 32′ No. it were required to find the Sun's true Azimuth, when his Altitude was 50° 00′ in the Afternoon. Firſt,

From the Com. of the Altitude 40° : 00′
Sub. the Com. of the Latitude 38 : 28
Natural V. Sine of the Difference ——— 1 : 32--,00035
Natural V. Sine of the Sun's diſt. from the Pole 67 : 54--,62377
,62342

The Log. of which Difference 62342 with the Square of the Radius is 29.794780
Coſine of the Latitude 51° 32′ —— 9.793831
Coſine of the Altitude 50 00 —— 9.808067
Sum ſubtract 19.601899
Remains 10.192882

Here the Remainder exceeds 10,000000; wherefore cancel the Characteriſtic 10, and the Number anſwering the remaining Logarithm is 1,5591; the Exceſs of which above Unity, *viz.* ,5591, gives the natural Sine of 34° 00′; whence the Sun's true Azimuth is North 124° 00′ Weſt. At which time, if the Sun's Magnetical Azimuth were North 110° 30′ Weſt, the Variation of the Compaſs would be 13° 30′ Weſt, as appears by the following Subtraction.

True Azimuth, North 124° : 00′ Weſt
Mag. Azimuth, North 110 30 Weſt
Variation 13 : 30 Weſt

N. B. If the Sun's Declination had been South, then the verſed Sine of the Sun's Diſtance from the elevated Pole, would have been equal to Unity plus the natural Sine of the Sun's Declination; which in Practice creates no more Trouble than when the Declination is North, if ſo much; ſince it is at leaſt as eaſy to take the natural Sine of an Arc, as to take the verſed Sine of its Complement to 90 Degrees; which Sines, and others with their reſpective Logarithms, *&c.* may readily be had out of *Sherwin*'s Mathematical Tables.

FINIS.

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